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Partial Transfers

by

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The transfer for compact fibrations has been studied for some time now, [1,2,3,4,6,7,8, 10], and has given rise to many applications [2,3,4,5,6]. One striking feature is that the transfer can be defined over a broad range of situations, namely fibrations with compact fibre, and has manifestations in all cohomology or homology theories. There are naturality formulas [4], [8] and a double coset formula [5,9] and others.

The purpose of this talk is to define transfers in special cases which do not have all of the apparatus of the transfer for fibrations alluded to above, and by so doing illuminate a basic principal upon which the transfer for fibrations stands and more important to apply these new transfers to obtain a new relationship among basic concepts.

Let $X \xrightarrow{f} Y$ be a map between two spaces. Let k be an integer. By a <u>transfer</u> for f with trace k, I shall mean a homomorphism $\tau : H_*(Y) \to H_*(X)$ such that the equation $f_* \circ \tau =$ (multiplication by k) is valid (for cohomology we require $\tau \circ f^* = k$). Since that equation is the only condition which I require of τ, I call such a transfer a

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"partial transfer".

Such a definition focuses attention on the equation $f_* \circ \tau$ = (mult. by k) = k. I regard this type of equation as important for the following philosophical reason: If A and B are abelian groups, there is only one homomorphism from A to B which we absolutely know exists, namely the zero homomorphism $0 : A \to B$. On the other hand, the only self homomorphisms $A \to A$ which we know always exist are the ones given by multiplication by an integer k. Now if we wish to say something about a homomorphism $f_*\ A \to B$ where A and B are not known, we are restricted to generalities such as f_* is onto or f_* is one to one, or there is a homomorphism $\tau : B \to A$ such that $f_* \circ \tau = k$ (or $\tau \circ f_* = k$). (Note that if f_* is an isomorphism, then the inverse isomorphism may be regarded as such a τ such that $f_* \circ \tau = 1$ and $\tau \circ f_* = 1$). Thus the transfer equation is one of the few ways there are for expressing information about a homomorphism in a general setting.

From now on we shall restrict ourselves to the following situation: we shall only consider smooth maps $f : M \to N$ where M and N are smooth closed manifolds of dimension m and n respectively. We shall assume M and N are oriented except when we use homology or cohomology with Z_2 coefficients. We shall let $[M] \in H_m(M)$ denote the fundamental class of M and $[\overline{M}]$ will denote the dual fundamental class in $H^m(M)$. The following fact will allow us to construct transfers.

Proposition 1: Suppose there is an $\alpha \in H_n(M)$ such that $f_*(\alpha) = k[N]$. Then f_* has a transfer of trace k (and f^* has a transfer of trace k.)

In fact we may define $\tau : H_*(N) \to H_*(M)$ by $\tau = (\cap\alpha) \circ f^* \circ (\cap[N])^{-1}$, and for cohomology we define $\bar{\tau} : H^*(M) \to H^*(N)$ by

$\bar{\tau} = (\cap[N])^{-1} \circ f_* \circ (\cap\alpha)$.

We may find, systematically, α's such that $f_*(\alpha) = k[N]$ by means of the following beautiful equation, which was shown to me by Andrew Casson. Suppose that $y \in N$ is a regular point of the map $f : M \to N$. By Sard's theorem, regular points are dense in N. Let $F = f^{-1}(y)$. So F is a closed manifold. Let $i : F \to M$ be the inclusion.

Proposition 2: $f^*([\overline{N}]) \cap [M] = i_*([F])$.

Proposition 2 will allow us to take information about a fibre of f and define transfers in a systematic way. We obtain the following theorem from propositions 1 and 2.

Theorem 3: There exists a transfer for $f : M \to N$ with trace equal to any Stiefel-Whitney number (in singular (co)homology with Z_2 coefficients) and there exists transfers for f with trace equal to any Pontrjagin number (in singular (co)homology with integer coefficients).

Now it is illegal to define a transfer without giving an application of it. The following corollary follows immediately from the existence of transfers with traces equal to Stiefel-Whitney numbers and the fact that a closed manifold bounds if and only if all its Stiefel-Whitney numbers are zero.

Corollary 4: *If* F *is not a boundary, then* $f^* : H^*(N;Z_2) \to H^*(M;Z_2)$ *is injective.*

This fact which relates such basic concepts appears not to have been observed before.

In the last part of this talk, we shall explain how transfers for fibre bundles with trace equal to the Euler-Poincare characteristics and Lefschetz numbers fit into the scheme of Proposition 1. In other words, we shall show how to find an α as in Proposition 1 such that $f_*(\alpha) = k[N]$ where k is $\chi(F)$ or a Lefschetz number. In this case we find α directly by using the transversality theorem instead of using Proposition 2. This is an older method of constructing the transfer which was never published since alternate methods were discovered soon after its discovery which were more amenable to constructing the transfer for fibrations in its full generality. Nevertheless, the old method is attractive from an intuitive point of view and so I shall take this occasion to insert it into the literature.

It is closely related to Dold's fixed point transfer construction [7,8].

§1. Proof of Proposition 1.

Recall we have $\alpha \in H_n(M)$ such that $f_*(\alpha) = k[N]$. We define $\tau : H_*(N) \to H_*(M)$ by $\tau = (\cap\alpha)\circ f^*\circ(\cap[N])^{-1}$. We must show that $f_*\circ\tau$ = multiplication by k. Let $\beta \in H_*(N)$. Then

$$\begin{aligned} f_*\tau(\beta) &= f_*\tau(x\cap[N]) = f_*(f^*(x)\cap\alpha) \\ &= x\cap f_*(\alpha) = x\cap(k[N]) = k(x\cap[N]) \\ &= k\beta \end{aligned}$$

where $x \in H^*(N)$ is defined by $x\cap[N] = \beta$.

For cohomology we define $\bar{\tau} : H^*(M) \to H^*(N)$ by $\bar{\tau} = (\cap[N])^{-1}\circ f_*\circ(\cap\alpha)$. Let $x \in H^*(N)$. Then

$$\begin{aligned} \bar{\tau}f^*(x) &= (\cap[N])^{-1}(f_*(f^*(x)\cap\alpha)) \\ &= (\cap[N])^{-1}(x\cap f_*(\alpha)) \\ &= (\cap[N])^{-1}(x\cap(k[N])) \\ &= k(\cap[N])^{-1}(x\cap[N]) = kx. \end{aligned}$$

§2. Proof of Proposition 2. $i^*([F]) = f^*([\bar{N}])\cap[M]$.

This beautiful fact is not as well known as it should be, and I do not know of a proof in the literature, so I shall include one here. A. Dold informs me that this proposition is a special case of a vague principal known to Hopf, which says that if a cycle and a cocycle are related by Poincare duality in N, then f^{-1}(cycle) and f^*(cocycle) should be related by Poincare Duality in M.

Let $* \in N$ be a regular point for $f : M \to N$. Then there exists a ball D about $*$ such that $f^{-1}(D)$ is diffeomorphic to $D \times F$, where, of course, $F = f^{-1}(*)$. Furthermore f restricted to $f^{-1}(D)$ acts like the projection $D \times F \to D$. Let $U \in H_m((D,D-*) \times F)$ and $\overline{V} \in H^n(D,D-*)$ be orientation classes. Then $U = V \times [F]$, hence

$$f^*(\overline{V}) \cap U = (\overline{V} \times 1) \cap (V \times [F]) = (\overline{V} \cap V) \times (1 \cap [F])$$

$$= 1 \times [F] \in H_{m-n}(D \times F).$$

Now the inclusion $(D,D-*) \times F \xrightarrow{i} (M,M-F)$ is an excision as is $i_1 : (D,D-*) \to (N,N-*)$. Using these excisions we identify $H_*((D,D-*) \times F)$ with $H_*(M,M-F)$ and $H^*(D,D-*)$ with $H^*(N,N-*)$. Let $i : D \times F \to M$, then

$$i_*([F]) = i_*(1 \times [F]) = i_*(f^*(\overline{V}) \cap U)$$

$$= ((i^*)^{-1}(f^*(\overline{V})) \cap i_*U)$$

$$= f^*((i_1^*)^{-1}(\overline{V})) \cap i_*U$$

Now $i_*U \in H_m(M,M-F)$ and $(i_1^*)^{-1}(\overline{V}) \in H^n(N,N-*)$ are generators and if $j : M \to (M,M-F)$ and $j_1 : N \to (N,N-*)$ are the inclusions, we have $j_*([M]) = i_*U$ and $j_1^*[(i_1^*)^{-1}(\overline{V})] = [\overline{N}]$

Hence $i_*([F]) = f^*((i_1^*)^{-1}(\bar{V})) \cap i_*U$

$$= f^*((i_1^*)^{-1}(\bar{V})) \cap j_*[M]$$

$$= j^*f^*((i_1^*)^{-1}(\bar{V})) \cap [M]$$

$$= f^*(j_1^*[(i_1^*)^{-1}(\bar{V})]) \cap [M]$$

$$= (f^*[\bar{N}]) \cap [M].$$

§3. Proof of Theorem 3.

Lemma 5: Suppose there exists an $x \in H^{m-n}(M)$ such that $i^*(x) = k[\bar{F}]$. Then there are transfers in homology and cohomology with trace k.

Proof: In view of Proposition 1, all we must show is that there is an $\alpha \in H_n(M)$ such that $f_*(\alpha) = k[N]$. Let $\alpha = (-1)^{n(m-n)} x \cap [M]$.

Then $k = k\langle[\bar{F}], [F]\rangle = \langle k[\bar{F}], [F]\rangle$

$$= \langle i^*(x), [F]\rangle = \langle x, i_*[F]\rangle$$

$$= \langle x, f^*([\bar{N}]) \cap [M]\rangle \quad \text{by Proposition 2}$$

$$= (-1)^{n(m-n)} \langle f^*[\bar{N}], x \cap [M]\rangle$$

$$= \langle f^*[\bar{N}], \alpha\rangle = \langle[\bar{N}], f_*(\alpha)\rangle$$

Hence $f_*(\alpha) = k[N]$ as was to be shown.

Lemma 6: If $k \in Z_2$ is a Stiefel-Whitney number of F or if $k \in Z$ is a Pontrjagin number of F, then there exists

<u>some</u> $x \in H^{m-n}(M)$ <u>such that</u> $i^*(x) = k[F]$.

Proof: Since F is the inverse image of a regular value, the normal bundle of F in M is a trivial bundle ε. Hence the tangent bundle τ_M of M restricted to F satisfies $(\tau_M|F) = \tau_F + \varepsilon$. Thus if k is a Stiefel-Whitney number (or a Pontrjagin number), then $k[F]$ is a Stiefel-Whitney class (Pontrjagin class) for the bundle τ_F and hence for the bundle $\tau_F + \varepsilon$ by the product formula. Since characteristic classes are natural, $k[F]$ must be the pull back of a characteristic class x of the bundle τ_M. Thus $k[F] = i^*(x)$.

Lemmas 5 and 6 immediately give us Theorem 3.

<u>Remark</u>: (1) If we try to apply the argument of Lemma 6 to the Euler class $\chi(\tau_F)$ of τ_F, which is $\chi(F)[F]$, we fail since $\chi(\tau_F) \neq \chi(\tau_F + \varepsilon)$. However, if we know that τ_F extends to a bundle α over M of the same dimension, then naturality would give us $i^*(x) = \chi(F)[F]$. If $f : M \to N$ is actually a fibre bundle, then the bundle of tangents along the fibre would give us the required α. Thus we have the transfer for fibrations in the special case of a fibre bundle where fibre, base and total spaces are smooth closed manifolds. This is the key case, for as shown in [6] every fibration with compact fibre and base is a "fibre retraction" of that case and hence, because of the nice properties of $\chi(F)$ and Lefschetz numbers the area of definition can be extended to fibrations.

(2) The above part of this talk arose from conversations with Andrew Casson, for which I am greatly indebted.

§4. Transfers and Transversality:

We shall be concerned with the following situation: let $F \to E \xrightarrow{p} B$ be a fibre bundle where F, E^m and B^n are smooth oriented closed manifolds. Suppose also that we have a map $f : E \to E$ such that $p \circ f = p$. Let $g = f|F$. Then in this case we shall show the existence of a transfer with trace Λ_g, the Lefschetz number of g. This will be done by showing the existence of an $\alpha \in H_n(E^m)$ such that $p_*(\alpha) = \Lambda_g[B]$. This will not be done by using the fibre inclusion $i : F \to E$ as in Proposition 2, instead we shall find a manifold M contained in E which represents α.

Let P be the pullback of $p : E \to B$ so that we have the following commutative diagram.

$$\begin{array}{ccc} P & \longrightarrow & E \\ \downarrow{\scriptstyle \bar{p}} & & \downarrow{\scriptstyle p} \\ E & \xrightarrow{p} & B \end{array}$$

Thus $P \subset E \times E$ is defined by $P = \{(e,e') \in E \times E \mid p(e) = p(e')\}$ and $\bar{p}(e,e') = e$.

Now consider the following two cross-sections, $\Delta : E \to P$ and $s : E \to P$. given by $\Delta(e) = (e,e)$ and $s(e) = (e,f(e))$.

Now $g : F \to F$ is homotopic to a map g' with a finite number of fixed points all of whose fixed point indicies are ± 1 and the sum of these adds up to Λ_g. Hence by the fibre homotopy extension property we know that f is fibre homotopic to a fibre preserving map f' which restricts on $p^{-1}(U) = U \times F$ to the map $1 \times g' : U \times F \to U \times F$ for some neighborhood U containing the point $p(F)$. Hence the cross-section s is homotopic to the cross-section s' given by $s'(e) = (e, f'(e))$. Furthermore $s'(E)$ is transverse to $\Delta(E)$ on the neighborhood $p^{-1}(U) \subset E$. Now we may isotopy s' relative to $p^{-1}(U)$ so that $s'(E)$ is transverse to $\Delta(E)$. Thus we obtain a closed manifold $M \subset \Delta(E) = E$ which is the intersection of $\Delta(E)$ and the isotopied $s'(E)$. Furthermore $M \cap p^{-1}(U)$ is the disjoint union of a finite number of copies of U. Thus M has the same dimension as B. Now M is a closed oriented n-dimensional submanifold of E and hence represents a homology class $[M] \in H_n(E)$. Consider the composition $q : M \subset E \xrightarrow{p} B$. Now q is a map of degree Λ_g because $q : (q^{-1}(U), q^{-1}(U) - q^{-1}(*)) \to (U, U-*)$ has degree Λ_g. Thus $q_*([M]) = \Lambda_g[B]$, hence $p_*(\alpha) = \Lambda_g[B]$ where α is $[M]$ included into E. Now Proposition 1 gives the required transfer.

Remark: We can give a construction of the transfer as an S-map at this point. The idea is to observe that $q : M \to B$

is a normal map, so it extends to a map between the Thom spaces of the normal bundle, $\bar{q} : M^{\nu} \to B^{\nu}$. Now using Spanier-Whitehead duality we know that M^{ν} is $n + \ell$ dual to M^+ (where M^+ is M union a point) and B^{ν} is $n + \ell$ dual to B^+ where ℓ is some integer. Hence there is an S-map $B^+ \to M^+$ which is dual to $\bar{q} : M^{\nu} \to B^{\nu}$. Hence for some integer k there is a map $\Sigma^k B^+ \to \Sigma^k M^+$. Now $\Sigma^k(B^+) \to \Sigma^k(M^+) \xrightarrow{\Sigma^k j} \Sigma^k E^+$ is the transfer where $j : M^+ \subset E^+$.

To see that q is a normal map, we shall show that $\tau_M = (p^* \tau_B)|M$. Let ν be the normal bundle of M in s(E). Let α be the normal bundle of $\Delta(E)$ in P. Since s(E) is transverse to $\Delta(E)$ we have $\nu = \alpha|M$. Now $\tau_E|M = \tau_M + \nu$. Also, since α (the normal bundle of $\Delta(E)$ in P) is equivalent to the bundle of tangents along the fibre of $p : E \to B$, we have $\tau_E = p^*(\tau_B) + \alpha$. Thus

$$(p^*(\tau_B)|M) + (\alpha|M) = \tau_E|M = \tau_M + \nu$$
$$= \tau_M + (\alpha|M)$$

Hence $p^*(\tau_B)|M = \tau_M$.

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