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ON REALIZING NAKAOKA'S COINCIDENCE POINT TRANSFER AS AN *S*-MAP

BY

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1. Introduction

In [6] the existence was asserted of a transfer for fibre bundles whose fibers were compact manifolds. This transfer is a homomorphism in singular homology or cohomology. In [1] this transfer was shown to be induced by an *S*-map, at least for fibre bundles with compact Lie groups as structure groups and finite complexes as base spaces. When a transfer is induced by an *S*-map then it exists in every cohomology and homology theory. In the case of the transfer in [1] the *S*-map led immediately to a simple topological proof of the Adams Conjecture.

The transfer map for finite covering spaces was known to be induced by an *S*-map. This fact was crucial in the proof of the celebrated Kahn-Priddy theorem [8].

At present there are several different transfers defined in various circumstances. Many are defined only for singular homology groups. Sometimes these are not realized by *S*-maps. For example the transfer for ramified coverings [10] does not commute with the Steenrod algebra [7] (also attributed to Dold). Hence it cannot be realized by an *S*-map. Recently, Ralph Cohen [4] has shown that the ramified covering transfer can be realized if the ramified covering is localized at certain primes. This was also known to Larry Smith.

In [9], Nakaoka obtained a transfer in the following situation: *Let*

$$E_1 \xrightarrow{p_1} B \quad \text{and} \quad E_2 \xrightarrow{p_2} B$$

be fibre bundles with closed oriented m-dimensional manifolds M_1 *and* M_2 *as fibres. Suppose that* $\pi_1(B)$ *acts trivially on both* $H^m(M_1; \mathbf{Z})$ *and* $H^m(M_2; \mathbf{Z})$. *Let* f *and* g: $E_1 \to E_2$ *be fibre preserving maps covering the identity* $1: B \to B$. *Then there is a homomorphism*

$$\tau_{f,g}: H^*(E_1; \mathbf{Z}) \to H^*(B; \mathbf{Z})$$

such that $\tau_{f,g} \circ p_1^*$ *is multiplication by* $\lambda(\bar{f}, \bar{g})$ *where* $\bar{f}$ *and* $\bar{g}$ *are the restrictions*

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of f and g to a fibre and $\lambda(\bar{f}, \bar{g})$ is the Lefschetz trace of $\bar{f}^*\bar{g}^!$, that is $\lambda(\bar{f}, \bar{g}) = \Sigma_i(-1)^i \mathrm{tr}(\bar{f}^*\bar{g}^!)_i$ where $g^!$ is the Umkehr map.

We shall show the following theorem.

THEOREM. *$\tau_{f,g}$ is induced by an S-map τ: $B \to E_1$ if B (and hence E_1 and E_2) is a closed oriented manifold, both fibre bundles p_1 and p_2 are smooth, and g is a normal map.*

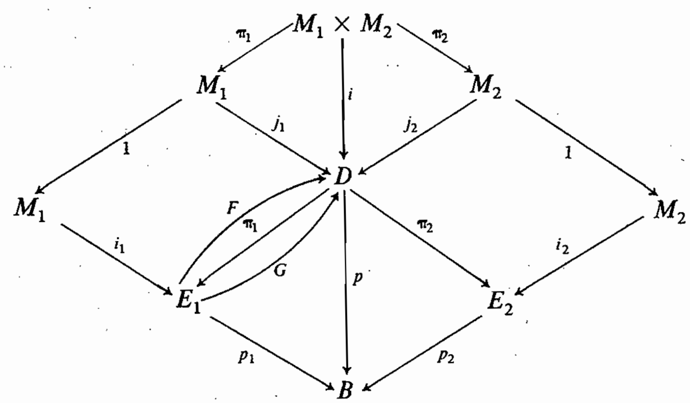
We require manifolds because we want to use Umkehr maps to define the transfer. We use smoothness to obtain bundles of tangents along the fibre easily.

Nakaoka's construction was a variation of the constructions in [6] and [2]. Our method here uses the techniques in [3]. The strategy is to define the transfer for singular cohomology as a composition of induced homomorphisms and Umkehr maps, in fact $p_1^! F^* G^!$ where F and G will be defined later. These maps are induced by maps of Thom spaces and then the condition that g is a normal map will yield a Thom space of a trivial bundle, which is a suspension. In order to express the transfer as $p_1^! F^* G^!$ we must give a somewhat different definition of the coincidence number transfer than Nakaoka's.

2. An S-map

Consider the following diagram of spaces and maps:

(1)



Here

$$E_1 \xrightarrow{p_1} B$$

is a fibre bundle with M_1 the fibre over a fixed b and

$$M_1 \xrightarrow{i_1} E_1.$$

Similarly

$$M_2 \xrightarrow{i_2} E_2 \xrightarrow{p_2} B$$

is a fibre bundle. Let D be the subspace of $E_1 \times E_2$ given by

$$D = \{(e_1, e_2) \in E_1 \times E_2 | p_1(e_1) = p_2(e_2)\}.$$

Let π_1: $D \to E_1$ be projection on the first factor, i.e., $\pi_1(e_1, e_2) = e_1$, and let π_2: $D \to E_2$ be projection on the second factor, $\pi_2(e_1, e_2) = e_2$. Then π_1 and π_2 are fibre bundles with fibres M_2 and M_1 respectively and the fibre inclusions are j_2 and j_1 respectively. The composition $p_1 \circ \pi_1 = p_2 \circ \pi_2 = p$ is itself a fibre bundle with fibre inclusion

$$M_1 \times M_2 \xrightarrow{i} D.$$

Now if f: $E_1 \to E_2$ is a fibre preserving map over the identity define the "graph" F: $E_1 \to D$ by

$$F(e_1) = (e_1, f(e_1)).$$

Similarly G is the graph of g: $E_1 \to E_2$. Note that

(2) $$\pi_1 \circ F = 1_{E_1}, \ \pi_2 \circ F = f \quad \text{and} \quad \pi_1 \circ G = 1_{E_1}, \ \pi_2 \circ G = g.$$

We now assume that E_1 and E_2 are smooth manifolds. Then D is a smooth manifold and all the bundles in (1) are smooth. Now let α_1 and α_2 be the bundle of tangents along the fibre

$$E_1 \xrightarrow{p_1} B \quad \text{and} \quad E_2 \xrightarrow{p_2} B$$

respectively. Also let β_1 and β_2 be tangents along the fibre for

$$D \xrightarrow{\pi_2} E_2 \quad \text{and} \quad D \xrightarrow{\pi_1} E_1$$

respectively. Then six pullback relations are easily seen to hold:

(3) $$\beta_1 = \pi_1^*\alpha_1, \quad \beta_2 = \pi_2^*\alpha_2,$$
$$F^*\beta_1 = \alpha_1, \quad F^*\beta_2 = f^*\alpha_2, \quad G^*\beta_1 = \alpha_1, \quad G^*\beta_2 = g^*\alpha_2.$$

For example, $F^*\beta_2 = F^*(\pi_2^*\alpha_2) = (\pi_2 \circ F)^*\alpha_2 = f^*\alpha_2$ by the equations in (2).

Now we imitate the approach of §5 of [3]. We embed E_1 in $\mathbf{R}^N$ by a smooth embedding k. Then we embed E_1 in $B \times \mathbf{R}^N$ by sending $e \mapsto (p_1(e), k(e))$. The normal bundle of E_1 in $B \times \mathbf{R}^N$ is $-\alpha_1$. Let $\hat{p}_1$: $B^N = \Sigma^N B^+ \to E_1^{-\alpha_1}$ be the collapsing map from the Thom space B^N of the trivial N dimensional bundle N to the Thom space of the normal bundle of E_1 in $B \times \mathbf{R}^N$.

The embedding F: $E_1 \to D$ extends to a bundle map $\alpha_1 \to \beta_1$. So F induces a map $\bar{F}$: $E_1^{-\alpha_1} \to D^{-\beta_1}$.

Finally $G\colon E_1 \to D$ is an embedding. Note that $G^*\beta_1 = \alpha_1$. Also note that the normal bundle of $G(E_1)$ in D is $G^*\beta_2$. Thus the normal bundle of $G(E_1)$ in $D^{-\beta_1}$ is $G^*(-\beta_1) \oplus G^*(\beta_2) = -\alpha_1 + g^*(\alpha_2)$. Thus if $\hat{G}$ denotes the collapsing map of $D^{-\beta_1}$ onto the Thom space of the normal bundle of $G(E_1)$ in $D^{-\beta_1}$ then $\hat{G}\colon D^{-\beta_1} \to E_1^{-\alpha_1+g^*(\alpha_2)}$.

The composition

$$\hat{G} \circ \bar{F}\hat{p}_1\colon \Sigma^N B^+ \to E_1^{-\alpha_1} \to D^{-\beta_1} \to E_1^{-\alpha_1+g^*(\alpha)} \tag{4}$$

gives rise to the required S-map

$$\tau\colon \Sigma^N B^+ \to \Sigma^N E_1^+$$

if $-\alpha_1 + g^*(\alpha_2)$ is stably trivial, i.e., $-\alpha_1 + g^*(\alpha_2) = N$.

Now $g^*(\alpha_2)$ is stably equivalent to α_1 if and only if $g\colon E_1 \to E_2$ is a normal map. This follows since

$$\begin{aligned} g^*(-\tau_{E_2}) &= g^*(-\alpha_2 - p_2^*(\tau_B)) \\ &= -g^*(\alpha_2) - g^*p_2^*(\tau_B) = -g^*(\alpha_2) - p_1^*(\tau_B) \end{aligned}$$

and

$$-\tau_{E_1} = -\alpha_1 - p_1^*(\tau_B)$$

and since $g^*(-\tau_{E_2}) = -\tau_{E_1}$ we see that $g^*(\alpha_2) = \alpha_1$. Thus we have shown:

Lemma 1. *If $g\colon E_1 \to E_2$ is a normal map then*

$$\tau = \hat{G} \circ \bar{F} \circ \hat{p}_1\colon \Sigma^N B^+ \to \Sigma^N E_1^+.$$

3. The induced homomorphism

First we shall show that $\tau\colon \Sigma^N B^+ \to \Sigma^N E_1^+$ induces $p_1^! F^* G^!$ on homology. More precisely we have the following lemma where

$$S\colon H^*(X) \to H^*(\Sigma^N X)$$

denotes the suspension isomorphism.

Lemma 2. $S^{-1}\tau^* S = \pm p_1^! F^* G^!$

Proof. Since p_1 and p_2 are orientable fibrations so are $\pi_1\colon D \to E_1$ and $\pi_2\colon D \to E_2$. Hence α_1, α_2, β_1 and β_2 are orientable vector bundle. Hence their associated Thom spaces admit Thom isomorphisms, which we shall denote by S. Then

$$\begin{aligned} S^{-1}\tau^* S &= S^{-1}\hat{p}_1^* \bar{F}^* \hat{G}^* S \quad \text{(by Lemma 1)} \\ &= \left(S^{-1}\hat{p}_1^* S\right)\left(S^{-1}\bar{F}^* S\right)\left(S^{-1}\hat{G}^* S\right) \\ &= \left(\pm p_1^!\right)(\pm F^*)(\pm G^!) \quad \text{(by (8), (9), (10) of §3 in [3])} \\ &= \pm p_1^! F^* G^! \end{aligned}$$

Now set $\tau_{f,g} = p_1^! F^* G^!$. We must show that $\tau_{f,g} \circ p_1^*$ is a multiplication by $\lambda(\bar{f}, \bar{g})$ where $\lambda(\bar{f}, \bar{g}) = \Sigma_i(-1)^i \operatorname{tr}(\bar{f}^* \bar{g}^!)_i$, the Lefschetz trace of $\bar{f}^*\bar{g}^!$.

Lemma 3. *$\tau_{f,g} \circ p_1^*$ is multiplication by k where $k = \langle i_1^* F^* G^!(1), [M_1]\rangle$*

Proof.

$$\begin{aligned} \tau_{f,g} p_1^*(x) &= p_1^! F^* G^! p_1^*(x) \\ &= p_1^! F^* G^!(G^*\pi_1^*) p_1^*(x) \\ &= p_1^! F^*\left(\pi_1^* p_1^*(x) \cup G^!(1)\right) \\ &= p_1^!\left(F^*\pi_1^* p_1^*(x) \cup F^* G^!(1)\right) \\ &= p_1^!\left(p_1^*(x) \cup F^* G^!(1)\right) \\ &= x \cup p_1^!\left(F^* G^!(1)\right). \end{aligned}$$

Now

$$p_1^!\left(F^* G^!(1)\right) \in H^0(B, \mathbf{Z}) \cong \mathbf{Z}$$

and so $\tau_{f,g} p_1^*$ is multiplication by $p_1^!(F^* G^!(1))$. But

$$p_1^!\left(F^* G^!(1)\right) = \langle i_1^*\left(F^* G^!(1)\right), [M_1]\rangle$$

Finally we must show that $\langle i_1^* F^* G^!(1), [M_1]\rangle = \lambda(\bar{f}, \bar{g})$, the Lefschetz trace of $\bar{f}^*\bar{g}^!$ where $\bar{f}$ and $\bar{g}$ are the restrictions of f and g to the fibre $M_1 \to M_2$.

Let $\bar{F}, \bar{G}\colon M_1 \to M_1 \times M_2$ be the graphs of $\bar{f}$ and $\bar{g}$ respectively, so that $\bar{F}$ and $\bar{G}$ are restrictions of F and G to the fibre M_1. Then $i \circ \bar{F} = F \circ i_1$ and $i \circ \bar{G} = G \circ i_1$ as can be seen by referring to diagram (1). By an argument similar to Lemma 1 of [3] we have that $\bar{G}^! i_1^* = \pm i^* G^!$. Thus

$$i_1^* F^* G^!(1) = \bar{F}^* i^* G^!(1) = \pm \bar{F}^* \bar{G}^!(i_1^*(1)) = \pm \bar{F}^* \bar{G}^!(1).$$

Thus we have shown:

Lemma 4. $i_1^* F^* G^!(1) = \pm \bar{F}^* \bar{G}^!(1)$.

Then the following lemma concludes the proof.

Lemma 5. $\bar{F}^* \bar{G}^!(1) = \lambda(\bar{f}, \bar{g})\overline{[M_1]} \in H^m(M_1; \mathbf{Z})$.

Proof. We will show this for rational coefficients **Q**, and that will imply the lemma. Let B be a homogeneous basis for $H^*(M_1, \mathbf{Q})$ and let $\hat{B}$ be a dual basis defined by $\hat{b} \cup a = \delta_{ba}[\overline{M_1}]$ where $a \in B$. Let Δ: $M_1 \to M_1 \times M_1$ be the diagonal map. Let μ be the unique element in $H^m(M_1 \times M_1; \mathbf{Q})$ defined by

$$\mu \cap [M_1 \times M_1] = \Delta_*[M].$$

Then $\mu = \Sigma_{b \in B}(-1)^{|b|}\hat{b} \times b$, (see p. 302, exercise 2 of [5]). Using the formula

$$(x \times y) \cap (\xi \times \eta) = (-1)^{|y||\xi|}(x \cap \xi) \times (y \cap \eta)$$

we obtain

$$\begin{aligned}\Delta_*[M_1] &= \mu \cap ([M_1] \times [M_1]) \\ &= \sum_{b \in B} (-1)^{|b|(1+m)}(\hat{b} \cap [M_1]) \times (b \cap [M_1])\end{aligned}$$

Note that $\overline{G} = (1 \times \bar{g}) \circ \Delta$. Hence

$$\begin{aligned}\overline{G}_*[M_1] &= (1 \times \bar{g})_* \Delta_*[M] \\ &= \sum_{b \in B} (-1)^{|b|(1+m)}((\hat{b} \cap [M_1]) \times (\bar{g}_*(b \cap [M_1]))\end{aligned}$$

Hence

$$\begin{aligned}\overline{G}^!(1) &= D^{-1}(\overline{G}_*[M_1]) \\ &= \sum_{b \in B} (-1)^{|b|(1+m)} D^{-1}((\hat{b} \cap [M_1] \times \bar{g}_*(b \cap [M_1])\end{aligned}$$

where D is the Poincaré Duality isomorphism. We have

$$\overline{G}^!(1) = \sum_{b \in B} (-1)^{|b|(m+1)} \cdot (-1)^{m|b|}\hat{b} \times D^{-1}\bar{g}_*(b \cap [M_1]).$$

Hence $\overline{G}^!(1) = \Sigma_{b \in B}(-1)^{|b|}\hat{b} \times \bar{g}^!(b)$. Then

$$\begin{aligned}\overline{F}^*\overline{G}^!(1) &= \Delta^*(1 \times \bar{f})^*\overline{G}^!(1) \\ &= \Delta^*\left(\sum_{b \in B} (-1)^{|b|}\hat{b} \times \bar{f}^*\bar{g}^!(b)\right)\end{aligned}$$

So $\overline{F}^*\overline{G}^!(1) = \Sigma_{b \in B}(-1)^{|b|}\hat{b} \cup \bar{f}^*\bar{g}^!(b)$. Now $\bar{f}^*\bar{g}^!(b) = \Sigma_{c \in B}\Gamma_b^c c$ where $\Gamma_b^c \in \mathbf{Q}$. Hence

$$\begin{aligned}\overline{F}^*\overline{G}^!(1) &= \sum_{b \in B} (-1)^{|b|}\hat{b} \cup \left(\sum_c \Gamma_b^c c\right) \\ &= \left(\sum_{b \in B} (-1)^{|b|}\Gamma_b^b\right)[M_1] \\ &= \lambda(\bar{f}, \bar{g})[M_1].\end{aligned}$$

Remark. The usual definition of the coincidences index which Nakaoka used is based on comparing two maps,

$$M_1 \xrightarrow{\Delta} M_1 \times M_1 \xrightarrow{f \times g} M_2 \times M_2$$

and

$$M_2 \xrightarrow{\Delta} M_2 \times M_2.$$

In our proof we compare

$$M_1 \xrightarrow{\Delta} M_1 \times M_1 \xrightarrow{1 \times f} M_1 \times M_2$$

with

$$M_1 \xrightarrow{\Delta} M_1 \times M_1 \xrightarrow{1 \times g} M_1 \times M_2.$$

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