ROBOTS AND FIBRE BUNDLES

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Dedicated to Professor Guy HIRSCH

Abstract. — An account of some topological insights in the designing of robot arms.

Topologists are unaccustomed to consulting on practical problems. Our discipline appears so removed from the applied branches of mathematics that it is always a surprise when a result plays a role in a non-theoretical environment. One day, William Fisher, an engineering doctoral student of Richard P. Paul at Purdue University asked me to participate on his oral exam committee. I agreed, and he explained his problem: Build a more flexible robot arm. It became clear that a topological viewpoint fit quite naturally into his problem and resulted in some interesting insights. After I discussed this application of topology with several colleagues, some encouraged me to write this paper. With that apology I begin.

A robot arm consists of links attached serially, one to the next. Two kinds of links are used, a prismatic link and a revolving link. The prismatic link moves in a straight line while the revolving link rotates about an axis. The straight lines and axes depend upon the configuration of the preceeding links in the arm. An example is shown of two links. The first rotates about its central axis. The second, hooked on to the first by a hinge, rotates about an axis through the hinge perpendicular to the plane formed by the two links.

Thus, each link's position relative to the preceeding link is given by one parameter, an angle in the case of a revolving link and a distance in the case of a prismatic link. Links made with more than one parameter, like

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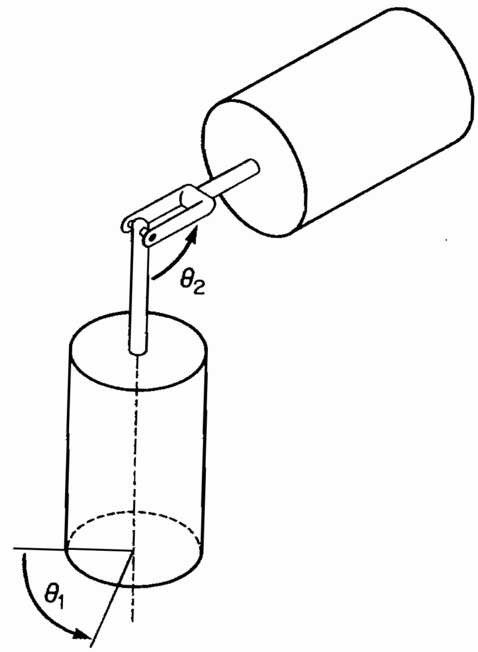

a universal ball joint, are not used because of considerations of mechanical power and control.

The problem is to design an arm which can track a moving frame through space. In other words, imagine a coordinate frame attached to the end of the arm. We want to program the arm to move so the coordinate frame follows any continuous path through space taking into account the orientation of the coordinate frame.

Robot arms with six links have been built. Six dimensions specify a coordinate frame in space, three for the origin and three for the orientation, thus the six linked arm should be able to position and orient itself into any coordinate frame desired in its work space. But the six-linked arm can move into certain singular positions so that there are nearby orientations which cannot be reached by a small adjustment of angles. Also the links of the arm spin around at great speeds and time is lost in the tracking.

We may explain what is occurring through topology. The position of an arm made of six revolving links is specified by six angles $(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6)$. Think of each angle as a point on a circle. The positions of the arm correspond to points on the six dimensional torus, T^6, which is the product of six circles $S^1 \times S^1 \times S^1 \times S^1 \times S^1 \times S^1$. The coordinate frame at the end of the arm at any given position represents a point in the space $\mathbb{R}^3 \times SO(3)$, where $\mathbb{R}^3$ is three dimensional Euclidean space and SO(3) is the space of rotations. This is the space of orthogonal 3×3 matrices of determinant $+1$. The correspondence from the position of the arm specified by $(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6)$ to the coordinate frame at the end of the arm is a smooth map f from $T^6 \to \mathbb{R}^3 \times SO(3)$.

Now f induces the derivative map, f_*, from the tangent bundle of T^6 to the tangent bundle of $\mathbb{R}^3 \times SO(3)$. Restricted to the fibre over b, f_* is a linear transformation between the two six dimensional vectorspaces over b and over $f(b)$. Most of the time it is an isomorphism, but sometimes, at singular points b, the Jacobian of f does not have maximal rank or equivalently f_* over b is *not onto* the fibre over $f(b)$. Since the tangent space at $f(b)$ is an approximation to a nearby neighborhood of $f(b)$, we see that slight changes of b will not give us all the coordinate frames near $f(b)$, at least not without infinite acceleration.

Fisher's idea was to add a seventh link to the arm. Then he would use the first three links to establish the origin and the last four links to get the orientation of the frame. Thus we have a map $f\colon T^4 \to SO(3)$. Unfortunately this map has singularities. In fact no matter how many links are used, there are still singularities, because every smooth map $f\colon T^n \to SO(3)$ must have singularities.

We call a map between two manifolds a *submersion* if there are no singular points. That is if its Jacobian always has maximal rank. A theorem of Ehresmann states that any submersion between two closed manifolds must be the projection of a fibre bundle [1]. Thus $f\colon T^n \to SO(3)$ must be a fibre bundle if f has no singular points. But this is impossible.

To see that $f\colon T^n \to SO(3)$ cannot be a fibre bundle we consider the universal covering $p\colon \mathbb{R}^n \to T^n$. Then $f \circ p\colon \mathbb{R}^n \to SO(3)$ is a fibre bundle. If F denotes a fibre of this bundle, then F must be a finite dimensional submanifold of $\mathbb{R}^n$. But since $\mathbb{R}^n$ is contractible, F is homotopy equivalent to the loop space $\Omega SO(3)$, which has homology in infinite dimensions.

So since singular points cannot be eliminated, they must be avoided. The extra dimension $f\colon T^4 \to SO(3)$ provided by the fourth link should give us some room to try and avoid the set of singular points S. We would like to have a procedure which will calculate angles $\vec{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4)$ for every orientation R so the arm with angles $\vec{\theta}$ will orient its coordinate frame in orientation R. Thus we want a function $\theta\colon SO(3) \to T^4$ so that $f(\theta(R)) = R$, and we want θ to be continuous. But this is impossible, for θ would be a cross-section to $f\colon T^4 \to SO(3)$, and such a cross-section cannot exist because the fundamental group $\pi_1(T^4) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ and $\pi_1(SO(3)) \cong \mathbb{Z}_2$ and $f_*\colon \pi_1(T^4) \to \pi_1(SO(3))$ cannot admit a cross-section.

We can try another tack. For every position of the arm $\vec{\theta}$ which is not singular, i.e. $\vec{\theta} \in T - S$, there is the orientation of the frame $f(\vec{\theta})$. We would like to calculate, for every nearby orientation $(f(\vec{\theta}))'$, a $\vec{\theta}'$ close to $\vec{\theta}$ so that $f(\vec{\theta}') = f(\vec{\theta})'$, that is a small change in the position of the arm which will result in the new orientation.

This can always be done. The pair $(\vec{\theta}, (f(\vec{\theta})'))$ can be approximated by the pair $(\vec{\theta}, \vec{v})$ where $\vec{v}$ is a vector in the tangent space of SO(3) at the point $f(\theta)$. Denote the set of all such pairs by f^*E and denote the total space of the tangent bundle of SO(3) by E. Then f^*E is the pullback of E over $T - S$ by $f\colon T - S \to SO(3)$. If D represents the total space of the tangent bundle of $T - S$ we have the following commutative diagram

$$\begin{tikzcd} f_*\colon D \arrow[r, "\alpha"] \arrow[d, "p"] & f^*E \arrow[r, "\beta"] \arrow[d, "p"] & E \arrow[d, "p"] \\ T - S \arrow[r, "1"] & T - S \arrow[r, "f"] & SO(3) \end{tikzcd}$$

where the p represent bundle projections and a and β are bundle maps whose composition $\beta \circ \alpha = f_*$ is the derivative map induced by f. What we want is a cross-section to α, call it $g\colon f^*E \to D$, so we would have $\alpha \circ g$ = identity. We can find an infinite number of such g because f_* is onto each fibre (because we removed the singular set S) and hence α is onto. The kernel of f_* is thus a one dimensional bundle and so f^*E is a direct summand of D.

Now the problem becomes one of choosing g so that $g(\vec{v})$ is not in the singular set for any small enough tangent vector $\vec{v}$. Fisher in [2] addresses this problem.

In connection with these considerations it is useful to calculate the axis of any rotation R. Every rotation in $\mathbb{R}^3$ has a unique axis of rotation, except for the identity rotation I of course. Richard P. Paul has a method which unfortunately involves two different cases. So one must first apply one formula to R and if that doesn't work one must apply a second formula. Fisher spent two months trying to find a single formula which would do the job. Unfortunately there cannot be such a formula. If there were, a continuous map would exist which took a rotation R into an equation $Ax + By + Cz = 0$ where the numbers (A, B, C) would be uniquely determined. Then the map ρ from $SO(3) - I$ to the space of lines through the origin, RP^2, would factor through $\mathbb{R}^3 - 0$. But $\rho\colon SO(3) - I \to RP^2$ is a homotopy equivalence and hence induces an isomorphism on the fundamental group $\pi_1(SO(3) - I) \underset{\cong}{\xrightarrow{\rho}} \pi_1(RP^2) \cong \mathbb{Z}_2$. Since $\pi_1(\mathbb{R}^3 - 0) \cong 0$, this is impossible.

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