Proceedings
1986 IEEE International Conference on Robotics and Automation.
Volume 3, pp. 1689-1691

ROBOTS AND TOPOLOGY

DANIEL H. GOTTLIEB

Dept. of math. , Purdue University.

ABSTRACT

Topology and topological invariants can be used to give insight into engineering problems. In the case at hand we consider a robot arm constructed out of links serially attached. Each link can move only by rotating about an axis through the previous link's point of contact with the link at hand. The problem is to devise an algorithm so that the end effector at the end of the arm can follow the motion of any frame through its workspace. The difficulty is that there are positions of the arm from which it cannot move instantaneously in the correct direction. These singular positions can be explained through topology. They can never be eleminated but they can be avoided.

1. Robot Arms.

The application of topology to practical problems will rarely yield precise formulas resulting in a solution. What one hopes for is a point of view resulting in greater understanding of the mathematical aspects of the problem. This viewpoint might give answers such as: This approach will not work; or these are the factors which will be present in a solution to the problem. At the least it should give a framework which integrates various componants of the problem into a coherant whole. In the discussion below, I believe the topological viewpoint realizes this hope.

The basic situation in topology is a continuous map of one topological space into another. We apply that to the case of a robot arm and the position and orientation of its end effector. Assume that the robot is made out of **n** links. Then the position of the arm is determined by specifying the **n** angles which each link makes with its predecessor. Now to a topologist, an angle is not a number, but a point on a circle **C** . So the arm's position is specified by a "vector" of **n** points on the circle; that is by an ordered tuple of **n** points on **C** . The set of all these vectors forms an n-dimensional torus **T** which is the cartesian product of **n** copies of **C**.
The two-dimensional torus for example is equal to **C × C** . This can be thought of as the surface of a donut. The torus **T** is the state space of the robot arm and will be the domain of our continuous mapping.

The range or target space is the cartesian product of $\mathbf{R}^3 \times \Theta$ where Θ denotes the space of orientations of the end effector. This is the space of n×n orthogonal matricies of determinant 1 producted with 3-dimensional Euclidean space. This can be seen by considering the end effector as a coordinate axis in three-space. Then the origin is a point in three space and the orientation of the axis is given by a rotation of the standard axis to the orientation of the end effector. This rotation is represented by a matrix which is orthogonal with determinant equal to 1.

The map **f** from **T** to $\mathbf{R}^3 \times \Theta$ is given by sending the position of the arm, which corresponds to a point on **T**, to the position and orientation of the end effector which corresponds to a point of $\mathbf{R}^3 \times \Theta$. This map **f** is clearly continuous. In fact it is differentiable as can be seen by imagining the impossibility of the smoothly moving arm leading to a jerky motion of the end effector. A representation of the function **f** as a formula can be given and is well known. This formula is necessary for any practical algorithm governing the motion of the arm. However in our analysis it is not necessary to know what the formula is.

Now let us consider the singular points of **f**. They are the points where the Jacobian matrix does not have maximal rank. That is, the singular points are those points where **f** does not map a nieghborhood onto a nieghborhood of its image. Mathematically this is not strictly true, but it is a good intuitive picture. We want **f** to be onto locally so that every instantaneous motion of the pursued frame can be duplicated instantaneously by the arm. actually, at the singular points the arm would need infinite accelarations to follow the frame in some directions.

We can easily see that no matter how many links are added to the arm, singular points must still exist. For simplicity, we consider the situation of $\mathbf{f}: \mathbf{T} \to \Theta$. That is a map of the position of the arm only to the orienatation of the end effector, neglecting the position of the end effector in three-space. If **f** has no singularities a theorem of Ehresmann states that the map **f** must be a fibre bundle, **[1]**. But a straight forward topological arguement shows that **f** cannot possibly be a fibre bundle. Therefore there must exist singularities. This result was shown by Hollerbach in **[3]** by using analytic arguements.

Another, simpler question can be decided using this point of view. It would be nice to be able to calculate a position of the arm for every orientation of the end effector. We would want our calculation algorithm to take nearby orientations to nearby arm positions. Expressed topologically, we want to find a mapping **s** from Θ to **T** so that the composition fs is the identity mapping and so that s is continuous. We call such an s a cross-section of f. Such a cross-section cannot exist by the following very typical topological arguement. Continuous maps of topological spaces induce homomorphisms between groups associated to those spaces. Such groups are fundemental groups, homotopy groups, homology groups, and cohomology groups. In the case at hand we will use fundemental groups. So f induces the homomorphism $f_*: \pi(T) \to \pi(\Theta)$ and s induces $s_*: \pi(\Theta) \to \pi(T)$. The composition $f_*s_* = (fs)_*$ = the identity homomorphism. Now $\pi(\Theta)$ = the group of order two. And $\pi(\Theta)$ equals a free group on n generators, which means that it has no torsion. Now since f_*s_* is the identity, s_* must be one to one. Thus the generator x of $\pi(\Theta)$, which satisfies $2x = 0$ is carried to a nonzero element in $\pi(T)$ which satisifies the equation $2s_*(x) = 0$. But since $\pi(T)$ has no torsion this equation implies that $s_*(x) = 0$ which contradicts the fact that s_* is one to one.

An arguement using the topological concepts of tangent bundle and vector bundle shows that an algorithm which uses the position of the arm and the small desired change in the orienatation of the end effector to calculate the new position of the arm can be made in a continuous fashion. Also it should be possible to make it avoid the singular points if the dimension of T exceeds that of $\Theta \times R^3$, that is to say if the number of links in the arm exceeds 6.

2. Rotations, Gears, and Pattern Recognition.

Other questions can be considered from a topological point of view. Every rotation in three-space has a well defined axis of rotation, except for the trivial rotation corresponding to the identity matrix. There are methods of calculating this axis given the matrix of the rotation, but they all use at least two formulas, one for each case depending on the value of some parameter. It would be nice to have only one unified formula for the calculation. This is impossible! Using topology we look at the problem from the point of view of continuous mappings. The map that associates to every rotation its axis is seen to be a continuous mapping f from the space of nontrivial rotations of three-space, call it A, to the space of lines through the origin P. The space P is well known. It is the projective space of dimension three. The map f is known to be a homotopy equivalence. This implies that the induced homomorphism f_*

on fundemental groups is one to one. Now if there were a single formula for converting rotations into axes in a continuous way, then there would be a map which would associate to each rotation a triple of numbers which could represent a point which lies on the axis of the rotation and is not the origin. Thus the map $\mathbf{f} = \mathbf{gh}$ where $\mathbf{h}: \mathbf{A} \to \mathbf{R}^3 - \mathbf{0}$ is the map of the previous sentance and $\mathbf{g}$ takes a point to the line through the origin and the point in three-space. Now $\pi(\mathbf{P})$ is the group of order two whereas the space $\mathbf{R}^3 - \mathbf{0}$ is simply connected, that is, it has trivial fundemental group. Thus again we have a contradiction stemming from the fact that $\mathbf{g}_*$ must be the trivial homomorphism and $\mathbf{f}_*$ must be one to one. More details can be found for everything above in **[2]**, but this paper was written for topologists.

Another possibility for a topological setting is the question of gears. In this situation a gear train made up of $\mathbf{n}$ gears has a state space of an n-dimensional torus. If the gears are out of round they exert a force at a specified point. The force vector is a three dimensional vector. So the assignment to the position of the gears of the force vector represents a function from the torus to three-space. If we think of a particular set of gears with interlocking teeth, the only motions through the state space possible now are paths called geodesics. At a certain speed the gear's motion is represented as a trajectory around the torus. The force then is a map of the real line, representing time, to three-space. When we disengage the gears and rotate them to a new position and engage them again we get a different geodesic. A more sophisticated setting which would take into account the change of force on the speed of the gear train would be given by a map from the tangent bundle of the torus to three-space.

Consider pattern recognition from the topological point of view. The pattern would be represented as a function between two topological spaces $\mathbf{f}: \mathbf{A} \to \mathbf{B}$. The problem would be to match $\mathbf{f}$ up with a standard function $\mathbf{F}$. The problem is that $\mathbf{F}$ may be distorted in various ways and so $\mathbf{f}$ and $\mathbf{F}$ may not agree on very many points of $\mathbf{A}$. We may think of the distortions as coming from deformations of $\mathbf{A}$ and $\mathbf{B}$. These deformations can be represented by homeomorphisms $\mathbf{h}: \mathbf{A} \to \mathbf{A}$ and $\mathbf{h}: \mathbf{B} \to \mathbf{B}$. Homeomorphisms are continuous maps with continuous inverses. Thus the problem becomes one of comparing $\mathbf{F}$ with $\mathbf{gfh}$ for some $\mathbf{f}$ and $\mathbf{g}$. We don't expect to find the $\mathbf{f}$ and $\mathbf{g}$. Instead we look for topological or geometric invariants which remain the same after composing by **f** and **g** . For example the number of maxima or critical points. If enough of these invariants agree for **f** and **F** , we would say they represented the same pattern.

References

1. C. Ehresmann, *Sur les espaces fibres differentiables.* C. R. Acad. Sci. Paris 224, 1611-1612 (1947).
2. D. H. Gottlieb, *Robots and fibre bundles.* Bulletin de la Societe Mathematique de Belguique, (to appear in the volume dedicated to G. Hirsch).
3. J. M. Hollerbach, *Optimal kinematic design for a seven degree of freedom manipulator.* 2nd International Symposium on Robotics Research, Kyoto, Japan, Aug. 20-23, 1984.