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Topology and the Robot Arm

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Abstract. A robot arm is in effect a smooth function from the space of positions of the arm to the space of positions of a coordinate frame attached to the end of the arm. For the most common robots built today, this means a map $f\colon T^n \to R^3 \times SO_3$. We describe the singularities of this map. The set of rotational singularities is the set of arm positions where the axes of the links are parallel to a plane. Thus, it is always two-dimensional. Also, we show that f is homotopic to a map which factors through a circle, and represents the generator of $\pi_1(SO_3)$. The engineering implication of these statements are discussed.

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1. Introduction

Every rotation in three-space can be described by specifying an axis and an angle of rotation about that axis. A very natural map $R\colon S^1 \times S^1 \times S^1 = T^3 \to SO_3$ arises as follows. For any three angles $(\theta_1, \theta_2, \theta_3) \in T^3$, we perform a rotation about the x-axis of angle θ_1 and then a rotation about the y-axis of angle θ_2 and then, finally, a rotation about the z-axis of angle θ_3. The resultant rotation is defined to be $R(\theta_1, \theta_2, \theta_3)$.

This is a smooth mapping. We shall determine S, the set of singularities of R, that is the set S of points $\theta \in T^3$ so that the Jacobian of R at θ does not have maximal rank. S consists of points of T^3 so that x, y' and z'' lie in the same plane; where y' is the image of the y-axis under the first rotation and z'' is the image of the z axis under the first two rotations.

We shall also show that R is homotopic to a map $T^3 \xrightarrow{\mu} S^1 \xrightarrow{\alpha} SO_3$ where μ is multiplication in the circle group and α represents the generator of $\pi_1(SO_3) \cong Z_2$.

In fact we shall consider maps $R\colon T^n \to SO_3$ given by specifying n axes of rotation $x_1, \ldots, x_n$ in R^3 and letting $R(\theta_1, \ldots, \theta_n)$ be the resulting rotation given by composing n rotations where the ith rotation is given by rotating an angle θ_i about the axis x_i.

We can build a physical representation of such a map. Consider a sequence of rigid rods, or links, l_i. Connect l_i to l_{i+1} by a point so that l_{i+1} can rotate about an axis y_{i+1} fixed on the link l_i. Thus, the axis y_1 is a fixed line and y_2 can be rotated about y_1, and y_3 can be rotated about y_2. The connected set of links form an arm. On the end of the arm we fix a coordinate frame. It will be convenient to let the x-axis of this frame be parallel to y_n, the axis about which l_n is joined to l_{n-1}.

The map $R: T^n \to R^3 \times SO_3 \to SO_3$ is given by taking a point in the torus $(\theta_1, \ldots, \theta_n)$, rotating each link l_i about the axis y_i through the angle θ_i and considering the frame on the end of the arm. Its origin is given by a point in R^3 and its orientation is a point in SO_3. Then we forget about the origin and consider only the orientation. We neglect, of course, possible collisions of the links with themselves.

The relationship of the axes $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ given by these two representations of R is a follows: $y_1 = x_1$, $y_2 =$ image of x_2 under the first rotation, $y_i =$ image of x_i under the composition of the first $i-1$ rotations.

2. Topological Theorems and Engineering Remarks

THEOREM 1. *A point $\theta \in T^n$ is a singular point for R if and only if the axes $y_1, \ldots, y_n$ are all parallel to a plane.*

Proof. To rotate a frame A into a nearby frame A' we would move the x axis of A into the x' axis of A' while letting the y, z plane rotate at a speed so that when x becomes x' we also have y becoming y' and z becoming z'. There is no trouble rotating the y, z plane about the moving axis x. But it may be impossible to instantaneousely move the x axis in the direction of the x' axis. Now note that the axes y_i and y_n, the last axis, lie in a plane. The rotation about y_i imparts an instantaneous velocity to y_n perpendicular to the plane. If we simultaneously rotate about all the y_i we impart to y_n a velocity which is the vector sum of all the velocities of y_n coming from the different rotations about the y_i. If all the y_i lie in a plane, the last axis y_n can only move perpendicular to that plane. On the other hand, if the axes do not lie in a plane, any instantaneous velocity vector can be created for y_n since the normals to the planes define by y_i and y_n span three space.

REMARKS. (a) Half this theorem appears in [5]. That is, if the axes are parallel to a plane, then the arm is in a singular position.

(b) Singular positions are bad for robot arms. They may lead to infinite accelerations to produce a desired slight change in the orientation of the end effector (i.e., frame). They are always there no matter how many links are used. The problem is to avoid them. They can be *avoided* by keeping the axes of the arm from becoming parallel to a plane. This simple observation is not widely known and does not seem to be in the literature. It should be noted by everyone who wants to study robot arms.

COROLLARY. *The set of singularities $S \subset T^n$ is two-dimensional.*

Proof. Assume $y_1, \ldots, y_n$ are parallel to a plane. Rotation about y_1 rotates the remaining axes and keeps them in a plane. Rotating the last link about y_n does not affect any axis, so they remain in a plane. Thus we see that any $\theta \in S$ is actually contained in a two-dimensional torus contained in S. If we rotate about ıte frame. It will be convenient to let axis about which l_n is joined to l_{n-1}. ven by taking a point in the torus e axis y_i through the angle θ_i and ı. Its origin is given by a point in R^3 en we forget about the origin and of course, possible collisions of the

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any interior axis y_i, we leave y_1 fixed and rotate y_{i+1} out of the plane defined by y_1 and y_i. So combinations of the interior axes can only be parallel to the plane defined by the first and last axes in discrete positions.

REMARKS. (a) For $n=3$, avoiding a singular point may be difficult. For some arms there are certain orientations of the end effector for which every position of the arm giving that orientation is a singular position. For $n>3$, the singularity sets can be avoided. This can be seen since the codimension $n-2$ is at least 2 and so the set S cannot cut T^n into two pieces locally.

(b) In [2] we show that any smooth map $f: T^N \to SO_3$ must have singularities by using a more sophisticated topological argument, but we do not know what the singularities look like in this more general case.

THEOREM 2. *A robot arm map $R: T^n \to SO_3$ is homotopic to the composition $T^n \xrightarrow{\mu} S^1 \xrightarrow{\alpha} SO_3$ where μ is the group multiplication of S^1 and α represents the generator of $\pi_1(SO_3) \cong Z_2$.*

Proof. A robot arm map is determined by the angles φ_i between the succeeding axes y_{i+1} and y_i. We can construct a homotopy of robot arms R_t by letting each angle φ_i become smaller and smaller until we end up with a robot arm R_1 whose axes are all parallel. No engineer would build such an arm, but it is seen that the x axis of the end effector is fixed and the y and z axes rotate about x through an angle which is the sum of the angle of rotation about each axis y_i. Thus $R_1: T^n \xrightarrow{\mu} S^1 \xrightarrow{\alpha} SO_3$ since the 'sum' means we use the group structure of S^1, and the fact that end effector for R_1 is specified only by the angle the y axis has rotated gives the S^1. Now we must show that α represents a generator of $\pi_1(SO_3)$. For this we consider the map $w: SO_3 \to S^2$ which takes a frame in R^3 to its x axis. (By a frame we mean an orthogonal set of unit vectors denoted x, y and z and called axes by abuse of language.) Then we have a fibre bundle $S^1 \to SO_3 \xrightarrow{w} S^2$. Now $w \circ R_1$ is a constant map whose image is x in S^2. Thus, α maps S^1 diffeomorphically onto the fibre. The homotopy exact sequence of this fibration is

$$\cdots \to \pi_1(S^1) \xrightarrow{\alpha_*} \pi_1(SO_3) \to \pi_1(S^2) \quad \text{or} \quad \cdots \to Z \xrightarrow{\alpha_*} Z_2 \to 0.$$

So the generator of Z maps under α_* to the generator of Z_2. This proves the result.

COROLLARY A. *$w \circ R: T^n \to S^2$ is homotopic to a constant.*

COROLLARY B. *There is no continuous cross-section to R or to $w \circ R$.*

Proof. A cross-section is a continuous map s so that $R \circ s$ = identity map. If there were a cross-section to R, then

$$\textit{identity} = (R \circ s)_* = R_* \circ s_*.$$

But since $\pi_1(T^n)$ has no torsion and $\pi_1(SO_3) \cong Z_2$ this is impossible. $(w \circ R)$ has no cross-section since otherwise S^2 would be contractible by Corollary A.

REMARKS. (a) The inverse kinematic problem of the engineers cannot be solved. That is, given an orientation of the end effector, calculate a configuration of the robot arm which yields that configuration. Since nearby orientations should be given by nearby configurations, we see that the inverse kinematic problem is equivalent to finding a continuous cross-section. This cannot be done.

(b) Corollary B was also known to Daniel Baker.

3. Avoiding Singularities and Inverse Kinematics

It was proposed on p. 770 of [1] to put functional restrictions on the joint angles to stay away from the singularities. That means find some map $\hat{f}: D \to T - S$ so that $R \circ \hat{f}$ has no singularities.

That procedure will not work if the domain D is a closed manifold. The map $R \circ \hat{f}$ must have singularities. This can be seen as follows.

Proof. $w \circ R \circ \hat{f}: D \to S^2$ is homotopic to a constant by Corollary A. If $R \circ \hat{f}$ has no singularities, neither will $w \circ R \circ \hat{f}$ since w is a fibre bundle projection. So by Ehresmann's theorem $w \circ R \circ \hat{f}: D \to S^2$ is a fibre bundle. But this is impossible since the fibre F must be a finite-dimensional manifold and also F is homotopy equivalent to $D \times \Omega S^2$ since $w \circ R \circ \hat{f}$ is contractible. Since ΩS^2, the loop space of S^2, has infinite-dimensional homology we have a contradiction.

We propose an approach here to avoid the singularities and solve the inverse kinematics problem at the same time. For every position of the arm θ and small change of end effector orientation $\Delta R(\theta)$, find a $\Delta\theta$ so that $R(\theta + \Delta\theta) = R(\theta) + \Delta R(\theta)$. (The plus signs are poetic license). That is, find a function depending on θ and $\Delta R(\theta)$ which gives $\Delta\theta$ so that the new arm position $\theta + \Delta\theta$ gives the new end effector position $R(\theta) + \Delta R(\theta)$.

We will convert this question to one of finding a cross-section to a function which represents these little changes. There is no topological difficulty finding the cross-section. This approach with more details is in [2].

Consider the commutative diagram

$$\begin{array}{rccccc} R_*: & D & \underset{\overset{\cdots}{s}}{\overset{\alpha}{\rightleftarrows}} & R^*E & \xrightarrow{\beta} & E \\ & \Big\downarrow p & & \Big\downarrow p & & \Big\downarrow p \\ & T - S & \xrightarrow[\iota]{} & T - S & \xrightarrow[R]{} & SO_3 \end{array}$$

where $E \xrightarrow{p} SO_3$ is the tangent bundle of SO_3, R^*E is the pullback to $T - S$ by the map R restricted to $T - S$ and $D \xrightarrow{p} T - S$ is the tangent bundle of $T - S$. All bundles are trivial bundles, although we will not use that fact here.

Now α is an onto bundle map and so $D \cong (\ker \alpha) \oplus R^*E$. Hence, there is a cross-section s to α.

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We may think of D as a space of pairs $(\theta, \Delta\theta)$ where $\Delta\theta$ represents a very small change in the robot arm and R^*E as a space of pairs $(\theta, \Delta R(\theta))$ where $\Delta R(\theta)$ represents a very small shift in the orientation of $R(\theta)$. Then α takes $(\theta, \Delta\theta)$ onto $(\theta, \Delta R(\theta))$ where $\Delta\theta$ results in the change in orientation $\Delta R(\theta)$. The cross-section s takes $(\theta, \Delta R\theta)$ into $(\theta, f(\theta, \Delta R\theta))$ where $f(\theta, \Delta R\theta)$ is a change in the arm which results in the change $\Delta R(\theta)$ of orientation.

One attempt using this approach is in [4].

REMARKS. Note that removing S not only removes the singularity of motion, but also permits the inverse kinematics problem to be solved in the above sense. With singular points we couldn't find a cross-section from the pullback R^*E to the tangent bundle of T^n.

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