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Spaces of Local Vector Fields

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This paper is dedicated to James Stasheff.

Abstract. Vector fields defined only over a part of a manifold give rise to indexes and to transfers. These local vector fields form a topological space whose relation to configuration spaces was studied by Dusa McDuff, and whose higher dimensional homotopy and homology promise invariants of parametrized families of local vector fields. We show that the assignment of the transfer to the vector field gives a map from the space of local vector fields of M into $Q(M^+)$ which stablizes into a homotopy equivalence.

1. Introduction

A *local vector field* on a smooth manifold M consists of an open subset U of $M - \dot{M}$ and a tangent vector field $\vec{u}$ on U which has compact zero set. For technical reasons we include as part of the definition the condition that for all $K \geq 0$, $\{x \epsilon U \mid |\vec{u}(x)| \leq K\}$ is compact.

The homotopy definition of the Hopf index of a local vector field $(U, \vec{u})$ is based on an embedding $M \subset R^s$ and realizes the index as the degree of a self map of the sphere S^s. This map has a factorization of the form $S^s \xrightarrow{\tau(U,\vec{u})} S^s \wedge M^+ \xrightarrow{\epsilon} S^s$, where ϵ collapses M to a point and $\tau(U, \vec{u})$ is the transfer associated to $(U, \vec{u})$. It is the fixed point transfer of Dold [**6**] as modified for vector fields [**2**].

Our main purpose here is to make more precise the relation between local vector fields and transfers. Let $\mathcal{V}(M)$ denote the set of local vector fields on M, let $\overline{\tau(M)}$ denote the fiberwise one point compactification of the tangent bundle $\tau(M)$ of M, and let $\mathrm{Sec}(M, \dot{M}, \overline{\tau(M)})$ denote the space of sections to $\overline{\tau(M)}$ whose restriction to the boundary $\dot{M}$ is the ∞–section. Compactification defines a bijection $\mathcal{V}(M) \to \mathrm{Sec}(M, \dot{M}, \overline{\tau(M)})$ and $\mathcal{V}(M)$ is given the induced topology. Then assigning to a local vector field its transfer defines a map $\lambda \colon \mathcal{V}(M) \to \mathrm{Map}(S^s; S^s \wedge M^+) \to Q(M^+)$. The method of [**3**] gives

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Theorem 1.1. *Let* $\dim(M) = n$. *Then* $\lambda\colon \mathcal{V}(M) \to Q(M^+)$ *is an* $(n-1)$–*equivalence.*

A stable result can be formulated by considering the space of local vector fields on $R^k \times M$, denoted by $\mathcal{V}^k(M)$ and letting $\mathcal{V}^\infty(M) = \overrightarrow{\lim}\ \mathcal{V}^k(M)$. The transfer construction leads to a map $\lambda^\infty\colon \mathcal{V}^\infty(M) \to Q(M^+)$ and we have

Corollary 1.2. $\lambda^\infty\colon \mathcal{V}^\infty(M) \to Q(M^+)$ *is a weak homotopy equivalence.*

These results have an equivariant as well as a fiberwise generalization which are discussed in sections 3 and 4. In section 5 we give a splitting of the space $\mathcal{V}_G(M)$ of G–local vector fields an a G–manifold M. We then show that in most cases G–local vector fields are classified by their transfer, as in the non equivariant case.

If V is a G–module with unit disk $D(V)$ and unit sphere $S(V)$, then $\mathcal{V}_G(D(V))$ is homeomorphic to $\Omega_V S^V$ and $\mathcal{V}_G(S(V))$ is homeomorphic to the space of G–self maps of $S(V)$, which we denote by $F_G(S(V))$. This provides a point of contact with well known results and methods. For example, the splitting of $\mathcal{V}_G(M)$ given in section 4, generalizes the splitting of $F_G(S(V))$ due to Petrie and tom Dieck [**5**] and Hauschild [**9**], and is based upon essentially the same geometric ideas.

2. Transfer construction

Recall that an *ex–space* E over B consists of a map $p\colon E \to B$ together with a preferred section $\Delta\colon B \to E$ [**7**]. If E and E' are ex–spaces over B and $A \subset B$, let $\mathrm{Map}_B(E, E \downarrow A; E')$ denote the space of ex–maps $f\colon E \to E'$ such that $f(e_a) = \Delta_{E'}(a)$ for every point e_a in the fiber over $a \varepsilon A$. If E is a fiber bundle over B let $\overline{E}$ denote its fiberwise one point compactification regarded as an ex–space with the section at infinity as preferred section. If Y is a pointed space we will often abbreviate the product ex–space $B \times Y$ to Y. If $\hat{E} \subset E$ are spaces over B then $E/_B\hat{E}$ denotes the ex–space obtained by identifying each fiber of $\hat{E}$ to a point.

Given an ex–space E over B and map $B \to B'$ we may regard $B \subset E$ as spaces over B' by composition. We will then write $T_{B'}(E)$ for $E/_{B'}B$. It is an ex–space over B'.

Now let M be a compact smooth manifold and fix an embedding $M \subset R^s$. Let ν be the normal bundle and $c\colon S^s \to T(\overline{\nu})/T(\overline{\nu} \downarrow \dot{M})$ the Pontryagin–Thom map. Let A be an open subset of M and let $\mathcal{V}(A)$ be the set of local vector fields $(U, \vec{u})$ such that $U \subset A$, topologized by means of the bijection

$$\mathcal{V}(A) \to \mathrm{Sec}(M, \dot{M} \cup (M - A);\ \overline{\tau(M)}).$$

Define

$$\lambda\colon \mathcal{V}(A) \to Q(A^+) \tag{2.1}$$

by

$$\begin{aligned}(2.2)\quad &\mathcal{V}(A) \to \mathrm{Sec}(M, \dot{M} \cup (M-A);\ \overline{\tau(M)}) = \mathrm{Map}_M(S^0 \times M, S^0 \times (\dot{M} \cup (M-A)); \overline{\tau(M)})\\ &\xrightarrow{\sigma} \mathrm{Map}_M(\overline{\nu}, \overline{\nu} \downarrow \dot{M} \cup (M-A); \overline{\tau(M) \oplus \nu}) = \mathrm{Map}_M(\overline{\nu}, \overline{\nu} \downarrow \dot{M} \cup (M-A); S^s \times M)\\ &\xrightarrow{T} \mathrm{Map}(T(\overline{\nu})/T(\overline{\nu} \downarrow \dot{M} \cup (M-A));\ S^s \wedge A^+) \xrightarrow{\mathrm{Map}(c)} \mathrm{Map}(S^s;\ S^s \wedge A^+) \longrightarrow Q(A^+),\end{aligned}$$

where σ is fiberwise suspension. Recall from [2] that the transfer of the local vector field $(\overline{U}, \vec{u})$ is the map

$$S^s \xrightarrow{c} T(\overline{\nu})/T(\overline{\nu} \downarrow \dot{M} \cup (M-A)) \xrightarrow{\varepsilon} T(\overline{\tau(M) \oplus \nu(M)} \downarrow A) = S^s \wedge A^+$$

where $\varepsilon(v_x) = \vec{u}(x) \oplus v_x,\ x \in U$, and $\varepsilon(v_x) = \infty,\ x \notin U$. Thus we see that the map λ simply assigns to a local vector field on A its transfer.

THEOREM 2.3. *Let $n = \dim(M)$. Let A be such that $M - A$ is a finite subcomplex of M. Then $\lambda\colon \mathcal{V}(A) \to Q(A^+)$ is an $(n-1)$–equivalence.*

PROOF. Let μ denote the duality map

$$S^s \xrightarrow{c} T(\overline{\nu})/T(\overline{\nu} \downarrow \dot{M} \cup (M-A)) \xrightarrow{\delta} T(\overline{\nu})/T(\overline{\nu} \downarrow \dot{M} \cup (M-A)) \wedge A^+$$

where $\delta(y_a) = y_a \wedge a$. It defines

$$D_\mu\colon \mathrm{Map}(T(\overline{\nu})/T(\overline{\nu} \downarrow \dot{M} \cup (M-A));\ S^s) \to \mathrm{Map}(S^s;\ S^s \wedge A^+) : D_\mu(f) = \mu(f \wedge 1)$$

Let

$$\alpha\colon \mathrm{Map}_M(\overline{\nu}, \overline{\nu} \downarrow \dot{M} \cup (M-A);\ S^s \times M) \to \mathrm{Map}(T(\overline{\nu}), T(\overline{\nu} \downarrow \dot{M} \cup (M-A));\ S^s)$$

denote the homeomorphism which sends f to $T(\pi f)$, π the projection $S^s \times M \to S^s$. In (2.2), $\mathrm{Map}(c)T$ is homotopic to $D_\mu \alpha$. Since D_μ is an $(n-1)$–equivalence for large s, the theorem follows. □

Let $\mathcal{V}^k(M)$ denote the set of local vector fields on $R^k \times M$ and for A open in M let $\mathcal{V}^k(A)$ denote the set of local vector fields $(U, \vec{u})$ for which $U \subset R^k \times A$. Compactification gives a bijection

$$\mathcal{V}^k(A) \to \mathrm{Map}_M(S^k \times M,\ S^k \times (\dot{M} \cup (M-A));\ \overline{R^k \oplus \tau(M)})$$

and $\mathcal{V}^k(A)$ is given the induced topology. Let $D = \dot{M} \cup (M-A)$ and define $\lambda^k\colon \mathcal{V}^k(A) \to Q(A^+)$ by

$$\begin{aligned}(2.4)\quad &\mathcal{V}^k(A) \to \mathrm{Map}_M(S^k \times M, S^k \times D;\ \overline{R^k \oplus \tau(M)}) \xrightarrow{\sigma}\\ &\mathrm{Map}_M((S^k \times M) \wedge_M \overline{\nu}, (S^k \times M) \wedge_M \overline{\nu} \downarrow D; \overline{R^k \oplus \tau(M) \oplus \nu}) \xrightarrow{T}\\ &\mathrm{Map}(S^k \wedge T(\nu)/S^k \wedge T(\nu \downarrow D); S^{k+s} \wedge A^+) \xrightarrow{\mathrm{Map}(c \wedge 1)} \mathrm{Map}(S^{k+s};\ S^{k+s} \wedge A^+) \to Q(A^+).\end{aligned}$$

Essentially the same argument as in the proof of (2.3) shows that λ^k is an $(n+k-1)$–equivalence.

Embed $R^k \times M$ into $R^{k+1} \times M$ by $(y_1, \dots, y_k, x) \to (0, y_1, \dots, y_k, x)$ and define $\mathcal{V}^k(A) \to \mathcal{V}^{k+1}(A)$ by $(U, \vec{u}) \to (R \times U, \vec{n} \times \vec{u})$ where $\vec{n}$ is the "outward" vector field on R defined by $\vec{n}(t) = t$. Let $\mathcal{V}^\infty(A) = \overrightarrow{\lim}\ \mathcal{V}^k(A)$. Then the λ^k define $\lambda^\infty\colon \mathcal{V}^\infty(A) \to Q(A^+)$.

Corollary 2.5. $\lambda^\infty\colon \mathcal{V}^\infty(A) \to Q(A^+)$ *is a weak homotopy equivalence.*

We close this section with a few simple observations about the space of local vector fields. For a disk D^n the canonical trivialization of the tangent bundle of D^n gives a homeomorphism $\mathcal{V}(D^n) = \Omega^n(S^n)$. So by considering disks of different dimension we see that $\mathcal{V}(M)$ is not a homotopy invariant of M. On the other hand it follows from (2.3) that if M and M' are homotopy equivalent, $\mathcal{V}(M)$ and $\mathcal{V}(M')$ have the same $(n-2)$–type where $n = \min\{\dim(M),\ \dim(M')\}$.

Consider now the different path components of $\mathcal{V}(M)$ when M is connected and $\dim(M) > 0$. They are indexed by the integers and, as Dusa McDuff [**10**] has shown, if M has non empty boundary the different path components are homotopy equivalent. Adding a collar and a local vector field on the collar of a prescribed index leads to homotopy equivalences between path components.

However if M is closed the different path components need not be homotopy equivalent. Consider the sphere S^n, $n \neq 1, 3, 7$. As we have noted, $\mathcal{V}(S^n) = F(S^n)$, the space of self maps of S^n. Consider the fibration $(\Omega^n S^n)_k \to F_k(S^n) \xrightarrow{\omega} S^n$ where ω is evaluation and $F_k(S^n)$ and $(\Omega^n S^n)_k$ are the components of degree k maps. We have that $\omega_*\colon \pi_n(F_0(S^n)) \to \pi_n(S^n)$ is onto since there is a section, while $\omega_*\colon \pi_n(F_1(S^n)) \to \pi_n(S^n)$ is not onto since S^n is not an H–space if $n \neq 1, 3, 7$. Now from the exact sequence of the fibration, we see that $\pi_{n-1}(F_0(S^n)) \neq \pi_{n-1}(F_1(S^n))$. Note however that by (2.3), the path components of $\mathcal{V}(M)$ have the same $(n-2)$–type where $n = \dim(M)$.

3. Equivariant and vertical vector fields

Suppose now that G is a compact Lie group, M is a compact smooth G–manifold, and A is an open G–invariant subset of M such that $M - A$ is a finite G–subcomplex. Then G acts on $\mathcal{V}(A)$ by $g(U, \vec{u}) = (gU, g\vec{u}g^{-1})$, $g \varepsilon G$. There is the space $Q(A^+, G) = \overrightarrow{\lim\limits_{V}}\ \mathrm{Map}(S^V, S^V \wedge A^+)$ the limit taken over the category of G–modules and G–monomorphisms. Fix an equivariant embedding of M into a G–module W and let $\lambda\colon \mathcal{V}(A) \to Q(A^+, G)$ denote the map of the previous section after replacing R^s by W. For $H \leq G$ let $n(H)$ be the minimum of the integers $\dim(A^K)$ and $\dim(A^J) - \dim(A^K)$, $J < K \leq H$, and K an isotropy subgroup of A. This defines a function n on subgroups of G.

Theorem 3.1. $\lambda\colon \mathcal{V}(A) \to Q(A^+, G)$ *is an* $(n-1)$*–equivalence. (That is,* $\lambda^H\colon \mathcal{V}(A)^H \to Q(A^+, G)^H$ *is an* $(n(H) - 1)$*–equivalence for each* $H \leq G$*.)*

The proof involves a straightforward generalization of the G–suspension theorem to a fiberwise version. Otherwise it is essentially the same as in the non equivariant case so we will omit the details.

Let $\mathcal{V}^W(A)$ denote the space of local vector fields on $W \times A$, W a G–module. We have $\lambda^W\colon \mathcal{V}^W(A) \to Q(A^+, G)$ defined by making the obvious changes in the definition of λ^k in (2.4). If $\phi: W \to \widetilde{W}$ is a G–monomorphism we have $\varphi: \mathcal{V}^W(A) \to \mathcal{V}^{\widetilde{W}}(A)$ by crossing with the outward vector field on $\widetilde{W}^\perp$. Let $\mathcal{V}^\infty(A, G) = \overrightarrow{\lim} \mathcal{V}^W(A)$ and $\lambda^\infty = \overrightarrow{\lim} \lambda^W$.

Corollary 3.2. *$\lambda^\infty: \mathcal{V}^\infty(A, G) \to Q(A^+, G)$ is a weak G–homotopy equivalence.*

In particular λ^∞ restricts to a weak homotopy equivalence of fixed point sets. Denote $\mathcal{V}^\infty(A, G)^G$ by $\mathcal{V}_G(A)$ and $Q(A^+, G)^G$ by $Q_G(A^+)$.

Corollary 3.3. *$\lambda^\infty: \mathcal{V}_G^\infty(A) \to Q_G(A^+)$ is a weak homotopy equivalence.*

We point out now how equivariant self maps of representation spheres, as studied in [**3**], fit into this general framework. Let V be a G–module and let $F_G(S(V))$ denote the space of G–self maps of the unit sphere $S(V)$ of V with the identity as base point. Representing $\tau(S(V))$ as the set of pairs $[x, x'] \varepsilon S(V) \times V$ such that $x \cdot x' = 0$, we have a base point preserving homeomorphism $F_G(S(V)) \to \mathcal{V}_G(S(V))$ by $f \to (U, \vec{u})$ where $U = \{x | f(x) \neq x\}$ and $\vec{u}(x) = [x, (1/(1 - x \cdot f(x)))(f(x) - (x \cdot f(x))x)]$. If G is finite and acts freely on V then $\mathcal{V}_G(S(V))$ is homeomorphic to $\mathcal{V}(S(V)/G)$ by $(U, \vec{u}) \to (U/G, \vec{u}/G)$. Thus we have a homeomorphism $F(S(V)) = \mathcal{V}(S(V)/G)$. The $(n-2)$–equivalence $F_G(S(V)) \to Q(S(V)/G^+)$ defined in [**3**] corresponds under this homeomorphism to $\lambda: \mathcal{V}(S(V)/G) \to Q(S(V)/G^+)$.

We will briefly describe the fiberwise generalization of the above construction, which relates the space of transfers of a smooth fiber bundle to its space of vertical local vector fields.

Let F be a compact smooth G–manifold and $\tilde{E} \to B$ a principal G–bundle, B a finite CW–complex. Let $E = \tilde{E} \times_G F$ and let $\tau(E \downarrow B)$ denote the bundle of tangents along the fiber, $\tau(E \downarrow B) = \tilde{E} \times_G \tau(F)$. Let $\dot{E} = \tilde{E} \times_G \dot{F}$. A *vertical local vector field* $(U, \vec{u})$ on $R^k \times E$ consists of an open set U of $R^k \times (E - \dot{E})$ and a section $\vec{u}: U \to R^k \oplus \tau(E \downarrow B)$ such that for $K \geq 0$, $\{e \varepsilon U |\, |\vec{u}(e)| \leq K\}$ is compact. Let $\mathcal{V}^k(E \downarrow B)$ denote the set of vertical local vector fields on $R^k \times E$ topologized by means of the bijection $\mathcal{V}^k(E \downarrow B) \to \mathrm{Map}_E(S^k \times E,\ S^k \times \dot{E};\ \overline{\tau(E \downarrow B)})$, and let

$$Q(E \downarrow B) = \overrightarrow{\lim}\ \mathrm{Map}_B(S^k \times B,\ S^k \wedge_B \overline{E}),$$

the *space of transfers* of the bundle $E \to B$.

Fix a G–embedding $F \subset W$, W a G–module, and a monomorphism $\tilde{E} \times_G W \to R^s \times B$. We then have

$$\tau(E \downarrow B) = \tilde{E} \times_G \tau(F) \to \tilde{E} \times_G W \times F \to R^s \times \tilde{E} \times_G F = R^s \times E.$$

Let ν be the orthogonal complement of $\tau(E \downarrow B)$ in $E \times R^s$. The embedding $E = \tilde{E} \times_G F \to \tilde{E} \times_G W \to B \times R^s$ has a Pontryagin–Thom map $c: S^s \times B \to T_B(\overline{\nu})/_B T_B(\overline{\nu} \downarrow \dot{E})$ which is an ex–map over B. Now define $\lambda^k: \mathcal{V}^k(E \downarrow B) \to$

$Q(\overline{E} \downarrow B)$ by

$$\begin{aligned}
&\mathcal{V}^k(E \downarrow B) \to \mathrm{Map}_E(S^k \times E,\ S^k \times \dot{E};\ \overline{R^k \oplus \tau(E \downarrow B)})\\
&\xrightarrow{\sigma} \mathrm{Map}_E(S^k \wedge_E \overline{\nu},\ S^k \wedge_E \overline{\nu} \downarrow \dot{E};\ \overline{R^k \oplus \tau(E \downarrow B) \oplus \nu})\\
&\xrightarrow{T_B} \mathrm{Map}_B(S^k \wedge_B T_B(\overline{\nu}),\ S^k \wedge_B T_B(\overline{\nu} \downarrow \dot{E});\ S^{k+s} \wedge_B \overline{E})\\
&\xrightarrow{\mathrm{Map}_B(1 \wedge_B c)} \mathrm{Map}_B(S^{k+s} \times B;\ S^{k+s} \wedge_B \overline{E}) \to Q(\overline{E} \downarrow B).
\end{aligned}$$

If $k = 0$, $\lambda = \lambda^0$ assigns to each vertical local vector field its transfer [**2**]. Again this transfer is Dold's fixed point transfer for fibre bundles [**6**] modified for vector fields. Let

$$\mathcal{V}^\infty(E \downarrow B) = \overrightarrow{\lim}\mathcal{V}^k(E \downarrow B),$$

and $\lambda^\infty = \overrightarrow{\lim}\lambda^k$.

Theorem 3.4. *$\lambda^\infty: \mathcal{V}^\infty(E \downarrow B) \to Q(\overline{E} \downarrow B)$ is a weak homotopy equivalence.*

A fiberwise duality map μ is given by

$$S^s \times B \xrightarrow{c} T_B(\overline{\nu})/_B T_B(\overline{\nu} \downarrow \dot{E}) \xrightarrow{\delta} T_B(\overline{\nu})/_B T_B(\overline{\nu} \downarrow \dot{E}) \wedge_B \overline{E},$$

where $\delta(y_e) = y_e \wedge e$. It induces

$$D_\mu: \mathrm{Map}_B(T_B(\overline{\nu}), T_B(\overline{\nu} \downarrow \dot{E});\ S^s \times B) \to \mathrm{Map}_B(S^s \times B;\ S^s \wedge_B \overline{E})$$

which is a $(t-1)$–equivalence if s is large relative to t [**1**]. The proof is now essentially the same as the proof of (2.3).

4. Splitting of $\mathcal{V}_G(M)$

Suppose that M is a compact G–manifold and H is a maximal isotropy subgroup. Let N be a G–tubular neighborhood of $M^{(H)} = G \times_{N(H)} M^H$ and identify N with the unit disk bundle of the normal bundle $\nu(M^{(H)} \subset M)$. Letting $p: N \to M^{(H)}$ denote the projection, we have $\tau(M) \downarrow N = p^*(\tau(M^{(H)})) \oplus \nu(M^{(H)} \subset M)$. Let $\vec{n}$ be the outward normal vector field on N, defined by $\vec{n}(v_y) = (1/1 - |v_y|)v_y$.

Lemma 4.1. *Let (X, A) be a finite CW pair on which G acts trivially, and $\Delta: N \times X \to \overline{\tau(M)} \downarrow N$ a G–lifting such that $\Delta | N \times A$ is given by $\Delta(v_y, a) = (v_y, \Delta(y, a) \oplus \vec{n}(v_y))$. Then Δ is G–homotopic relative to $N \times A$ to Δ' such that $\Delta'(v_y, x) = (v_y, \Delta(y, x)) \oplus \vec{n}(v_y))$.*

This is an easy consequence of the G–covering homotopy extension property. Now let $\tilde{p} = p|\mathrm{Int}(N)$ and let $M_0 = M - \mathrm{Int}(N)$. Define

$$\zeta: \mathcal{V}_G(M_0) \times \mathcal{V}_G(M^{(H)}) \longrightarrow \mathcal{V}_G(M) \tag{4.2}$$

by $\zeta((U, \vec{u}), (W, \vec{w})) = (U, \vec{u}) \sqcup (\tilde{p}^{-1}(W), \tilde{p}^*(\vec{w}) \oplus \vec{n})$

LEMMA 4.3. *ζ is a weak homotopy equivalence.*

PROOF. Let X be a finite CW–complex on which G acts trivially. We wish to show that

$$\zeta_{\#}\colon [M_0 \times X, \dot{M}_0 \times X; \overline{\tau(M_0)}]^G_{M_0} \times [M^{(H)} \times X,\ \dot{M}^{(H)} \times X,\ \overline{\tau(M^{(H)})}]^G_{M^{(H)}} \longrightarrow [M \times X,\ \dot{M} \times X;\ \overline{\tau(M)}]^G_M$$

is bijective. To show that $\zeta_{\#}$ is onto let $\Delta\colon M \times X \to \overline{\tau(M)}$ be a G–map over M. By the previous lemma we may assume that $\Delta|N \times X$ is given by $\Delta(v_y, x) = (v_y, \Delta(y,x) \oplus \vec{n}(v_y))$. Let $f = \Delta|M_0 \times X$ and $f' = \Delta|M^{(H)} \times X$. Then $\Delta = \zeta(f, f')$. A similar argument shows that $\zeta_{\#}$ is one–one. $\square$

If H is an isotropy subgroup of M and N is a G–tubular neighborhood of $M_{(H)}$ in M, define $\alpha_H\colon \mathcal{V}_{W(H)}(M^H) \to \mathcal{V}_G(M)$ by

$$\alpha_H(W, \vec{w}) = (\tilde{p}^{-1}(G \times_{N(H)} W),\ \tilde{p}^*(1 \times_{N(H)} \vec{w}) \oplus \vec{m}).$$

THEOREM 4.4. *There is a weak homotopy equivalence $\psi\colon \underset{(H)}{\times} \mathcal{V}_{W(H)}(M_H) \to \mathcal{V}_G(M)$, the product taken over the conjugacy classes of isotropy subgroups of M, such that for each (H), $\psi|\mathcal{V}_{W(H)}(M_H) \simeq \alpha_H$.*

PROOF. If H is a maximal isotropy subgroup then $\mathcal{V}_{W(H)}(M_H) = V_G(M_{(H)})$ by $(W, \vec{w}) \to (G \times_{N(H)} W,\ 1 \times_{N(H)} \vec{w})$ and $M_{(H)} = M^{(H)}$. Replacing $\mathcal{V}_G(M^{(H)})$ in (4.2) by $\mathcal{V}_{W(H)}(M_H)$ gives a weak homotopy equivalence

$$\psi'\colon \mathcal{V}_G(M_0) \times \mathcal{V}_{W(H)}(M_H) \to V_G(M)$$

such that $\psi'|V_{W(H)}(M_H) = \alpha_H$. We may assume inductively that the theorem holds for M_0. Note that the canonical strong deformation retraction of $M - M^{(H)}$ to M_0 restricts to a strong deformation retraction of M_K to $(M_0)_K$, for each isotropy subgroup K such that $(K) \neq (H)$. It follows easily from this that the inclusion $V_{W(K)}((M_0)_K) \to V_{W(K)}(M_K)$ is a homotopy equivalence. This completes the proof. $\square$

5. Classification

It follows from theorem (2.3) that $\lambda\colon \pi_0(\mathcal{V}(M)) \to \omega_0(M^+)$ is a bijection if $\dim(M) > 1$, and it is easily checked that λ is also bijective if $\dim(M) = 1$. If M is 0–dimensional then λ is injective. (There are two local vector fields on a point — the empty one with index 0 and the non empty one with index 1.) In other words, local vector fields are classified up to homotopy by their transfer. An inductive proof of this classification is given in [**8**]. Similar observations based on (3.1) give the following.

LEMMA 5.1. *Let M be a compact G–manifold and A an open subset of M on which G acts freely. Then $\lambda\colon \pi_0(\mathcal{V}_G(A)) \to \omega^G_0(A^+)$ is bijective if* $\dim(M) > 0$ *and injective if* $\dim(M) = 0$.

Theorem 5.2. *Let M be a G–manifold such that for each pair of isotropy subgroups $H < K$, $\dim(M^K) + 1 < \dim(M^H)$. Then $\lambda\colon \pi_0(\mathcal{V}_G(M)) \to \omega_0^G(M^+)$ is injective.*

Proof. Suppose that $\lambda(U, \vec{u}) = \lambda(V, \vec{v})$. Now from the splitting of $V_G(M)$ we may assume that $(U, \vec{u}) = \bigcup_{(H)} \alpha_H(U'_H, \vec{u}'_H)$ where $(U'_H, \vec{u}'_H) \varepsilon \mathcal{V}_{U(H)}(M_H)$, and similarly that $(V, \vec{v}) = \bigcup_{(H)} \alpha_H(V'_H, \vec{v}'_H)$ where $(V'_H, \vec{v}'_H) \varepsilon V_{W(H)}(M_H)$. Fix an isotropy subgroup H. If $(K) \neq (H)$ then since the zeroes of $\alpha_K(U'_K, \vec{u}'_K)$ lie in $M_{(K)}$, $\alpha_K(U'_K, \vec{u}'_K)^H$ is a non zero vector field and is therefore $W(H)$–homotopic to the empty vector field. Consequently, $(U, \vec{u})^H \underset{W(H)}{\simeq} (U'_H, \vec{u}'_H)$ and similarly $(V, \vec{v})^H \underset{W(H)}{\simeq} (V'_H, \vec{v}'_H)$. We now have, upon restricting to M^H that $\lambda(U'_H, \vec{u}'_H) = \lambda(V'_H, \vec{v}'_H)$ in $\omega_0^{W(H)}(M^{H+})$. By our assumption on M we see that $\omega_0^{W(H)}(M_H^+) \to \omega_0^{W(H)}(M^{H+})$ is injective. Therefore, $\lambda(U'_H, \vec{u}'_H) = \lambda(V'_H, \vec{v}'_H)$ in $\omega_0^{W(H)}(M_H^+)$ and by the lemma, $(U'_H, \vec{u}'_H) \simeq (V'_H, \vec{v}'_H)$ in M_H. This completes the proof. □

Corollary 5.3. *Let M be a compact G–manifold such that* $\dim(M^K) + 1 < \dim(M^H)$ *for each pair $H < K$ of isotropy subgroups. Two G–local vector fields $(U, \vec{u})$ and $(V, \vec{v})$ on M are G–homotopic if and only if for each isotropy subgroup H and component C of M^H, Ind $(U^H, \vec{u}^H)|C =$ Ind $(V^H, \vec{v}^H)|C$.*

Proof. If follows from the Segal–tom Dieck splitting theorem [**4**] that the map $\omega_0^G(M^+) \to \sum_{(H)} \omega_0(M^{H+})$ obtained by restricting to fixed point sets is injective. □

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