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A CERTAIN SUBGROUP OF THE FUNDAMENTAL GROUP.

By D. H. Gottlieb.

Introduction.[1] Let X be a topological space with x_0 as a base point A homotopy $H: X \times I \rightarrow X$ is called a *cyclic homotopy* if

$$H(x,0) = H(x,1) = x.$$

In another notation, h_t is a cyclic homotopy if $h_0 = h_1 = 1_X$, where 1_X denotes the identity map of X.

If h_t is a cyclic homotopy, the path given by $\sigma: I \rightarrow X$ such that $\sigma(s) = h_s(x_0)$ will be called the *trace* of h_t. The trace is obviously a closed path.

The set of homotopy classes of those loops which are the trace of some cyclic homotopy form a subgroup of the fundamental group which we shall denote by $G(X, x_0)$. It is the purpose of this paper to study $G(X, x_0)$, establish some elementary properties, compute it for one dimensional graphs, two dimensional compact manifolds, lens spaces and projective spaces. In addition its effects on the universal covering space and the mapping space X^X will be discussed.

Jaing Bo-Ju, in a recent paper [1], has also investigated this group. He was mostly interested in the role the group played in the Nielsen-Wecken theory of fixed point classes. Some properties of $G(X, x_0)$ proved here were mentioned by Jaing Bo-Ju, but they were not of the same generality except in the cases of Theorem 1.8 and Theorem II.4.

The present paper is divided into four parts. The first part deals with the elementary properties of $G(X, x_0)$, and $G(X, x_0)$ is computed for many kinds of spaces. In particular, Corollary I.13 tells us that if X is aspherical, then $G(X, x_0)$ is the center of $\pi_1(X, x_0)$.

In the second section, the role of $G(X, x_0)$ as the subgroup of the group of deck transformations of the universal covering space is discussed, leading to the calculation of $G(X, x_0)$ for lens spaces and for projective spaces. Theorem II.7 gives a condition for a homeomorphism to be in the center of a discrete group of homeomorphisms acting freely on a contractible space.

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In part III, the relation of $G(X, x_0)$ to the mapping space X^X is discussed. If X is aspherical, it is shown that the identity component of X^X has the homotopy groups, $\pi_1(X^X, 1_X) \cong Z(\pi_1(X, x_0))$, the center of $\pi_1(X, x_0)$, and $\pi_n(X^X, 1_X) = 0$ for $n > 1$. M.-E. Hamstrom has investigated the homotopy groups of the space of homeomorphisms of two dimensional aspherical manifolds. They turn out to be the same as the homotopy groups of the space of mappings. This indicates a deeper relation between the mapping space and the subspace of homeomorphisms of manifolds.

In part IV, Theorem IV. 1 says that $G(X, x_0) = 1$ if X is compact and the Euler Poincaré number is zero. This fact may be of some use in computing Homeotopy groups; see G. S. McCarty [5]. Also we have Corollary IV. 3 which says that if $\chi(X) \neq 0$ and X is aspherical, then $Z(\pi_1(X)) = 1$. This result is applied to subcomplexes X of S^n and yields facts about $S^n - X$.

I. The group $G(X, x_0)$.

§ 1. $G(X, x_0)$.

We shall concern ourselves only with pathwise connected *C. W.*-complexes in this paper. Let X be one such with x_0 as a base point. We begin our investigation by inquiring; which loops are the trace of some cyclic homotopy? The first theorem shows that the answer depends only on the homotopy classes of the loops.

If σ is a loop. i. e., $\sigma \colon I \to X$ such that $\sigma(0) = \sigma(1) = x_0$, then $[\sigma]$ shall denote the equivalence class of all loops α homotopic to σ under a homotopy h_t such that $h_t(0) = h_t(1) = x_0$. In symbols, this will be written $\alpha \cong \sigma \operatorname{rel} x_0$. We shall also regard σ as a map from the circle (S^1, s_0) to (X, x_0) and $[\sigma]$ will denote the set of all α such that $\alpha \cong \sigma \operatorname{rel} x_0$.

THEOREM I. 1. *If σ is the trace of a cyclic homotopy, and $\alpha \in [\sigma]$, then α is the trace of a cyclic homotopy.*

Proof. Let $H \colon X \times I \to X$ be a cyclic homotopy with σ as its trace and let h_t be the homotopy connecting σ with α. Let L be the subcomplex of $X \times I$ given by $(X \times 0) \cup (X \times 1) \cup (x_0 \times I)$. Define a partial homotopy of H on L as follows: $k_s \colon L \to X$ such that $k_s(x, t) = x$ if $t = 0$ or $t = 1$ and $k_s(x_0, t) = h_s(t)$.

Now L is a sub complex of $X \times I$, and hence has the homotopy extension property. This means that there is a homotopy $K_t \colon X \times I \to X$ such that $K_0 = H$ and $K_t \mid L = k_t$. Then $K_1 \colon X \times I \to X$ is a cyclic homotopy on X with trace α.

Definition. Let $G(X, x_0)$ be the set of all elements $[\sigma] \in \pi_1(X, x_0)$ such that σ is the trace of a cyclic homotopy on X.

THEOREM I. 2. *$G(X, x_0)$ is a subgroup of $\pi_1(X, x_0)$.*

Proof. Let $[\alpha]$ and $[\beta] \in G(X, x_0)$. Let h_t and k_t be the required cyclic homotopies respectively. Define a homotopy $l_t \colon X \to X$ such that $l_t(x) = h_{2t}(x)$ for $0 \leqq t \leqq \frac{1}{2}$ and $l_t(x) = k_{2t-1}(x)$ for $\frac{1}{2} \leqq t \leqq 1$. The trace of l_t is the loop $\alpha \cdot \beta$. Hence $[\alpha \cdot \beta] = [\alpha] \cdot [\beta] \in G(X, x_0)$.

Also $[\alpha]^{-1} \in G(X, x_0)$ since $[\alpha]^{-1} = [\alpha^{-1}]$ and α^{-1} is the trace of $h_{1-t}\colon X \to X$.

The next theorem shows that $G(X, x_0)$, viewed as a subgroup of $\pi_1(X, x_0)$ is independent of the choice of the base point x_0. Because of this, we shall abbreviate $G(X, x_0)$ to $G(X)$ when no confusion occurs.

Let $\sigma \colon I \to X$ be a path such that $\sigma(0) = x_0$ and $\sigma(1) = x_1 \in X$. Then σ induces an isomorphism $\sigma_* \colon \pi_1(X, x_1) \cong \pi_1(X, x_0)$ such that $\sigma_*([\alpha]) = [\sigma \cdot \alpha \cdot \sigma^{-1}]$.

THEOREM I. 3. *$\sigma_* \colon G(X, x_1) \cong G(X, x_0)$.*

Proof. Since σ_* is 1-1, all we must show is that $\sigma_*(G(X, x_1)) \subseteq G(X, x_0)$.

Let $[\alpha] \in G(X, x_1)$. Then there exists a cyclic homotopy $H \colon X \times I \to X$ with trace α. By the homotopy extension property, there is homotopy $J \colon X \times I \to X$ such that $J(x, 0) = x$ and $J(x_0, t) = \sigma(t)$.

Define $K \colon X \times I \to X$ by

$$\begin{aligned} K(x,t) &= J(x, 3t), && 0 \leqq t \leqq \tfrac{1}{3} \\ K(x,t) &= H(J(x,1), 3t-1), && \tfrac{1}{3} \leqq t \leqq \tfrac{2}{3} \\ K(x,t) &= J(x, 3(1-t)), && \tfrac{2}{3} \leqq t \leqq 1. \end{aligned}$$

Now K is a cyclic homotopy and its trace with respect to x_0 is $\sigma \cdot \sigma \cdot \sigma^{-1}$. So $\sigma_*[\alpha] = [\sigma \cdot \alpha \cdot \alpha \sigma^{-1}] \in G(X, x_0)$.

§ 2. $P(X, x_0)$ And Computations.

We now establish some notation. Suppose (X, x_0) and (Y, y_0) are two spaces with base points, then we will always assume that $X \times Y$ has the base point (x_0, y_0). Also, X will denote $X \times y_0$ and Y will denote $x_0 \times Y$ and $X \vee Y = (X \times y_0) \cup (x_0 \times Y)$.

Remark I. Let $\sigma \colon (S^1, s_0) \to (X, x_0)$. Then $[\sigma] \in G(X, x_0)$ if and only if the map $f \colon X \vee S^1 \to X$ such that $f(x) = x$ whenever $x \in X$ and $f(s) = \sigma(s)$ if $s \in S^1$ can be extended to $X \times S^1$.

The elements of $\pi_1(X, x_0)$ operate on $\pi_n(X, x_0)$ as a group of automorphisms in a standard way, [4].

Definition. The set of elements of $\pi_1(X, x_0)$ which operate trivially on all $\pi_n(X, x_0)$ form a subgroup which will be denoted as $P(X, x_0)$.

Remark II. $[\alpha] \in \pi_1(X, x_0)$ operates trivially on $\pi_n(X, x_0)$ if and only if for every map $f\colon S^n \to X$, there exists an extension $F\colon S^n \times S^1 \to X$ such that $F \mid S^1 = \alpha$.

Now we are in position to prove the next theorem, whose corollaries will give us $G(X)$ for many spaces.

THEOREM I. 4. $G(X, x_0) \subseteq P(X, x_0)$.

Proof. Let $[\alpha] \in G(x, x_0)$. By Remark I, we have a map $H\colon X \times S^1 \to X$ such that $H \mid X = 1_x$ and $H \mid S^1 = \alpha$. Let $f\colon (S^n, r_0) \to (X, x_0)$ be any map from an n-sphere to X. We shall define a map $F\colon S^n \times S^1 \to X$ as follows; $F(r, s) = H(f(r), s)$ for $r \in S^n$ and $s \in S^1$. Since $F(r, s_0) = H(f(r), s_0) = f(r)$, $F \mid S^n = f$. Also $F(r_0, s) = H(x_0, s) = \alpha(s)$ implies that $F \mid S^1 = \alpha$. Therefore, by Remark II, $[\alpha] \in P(X, x_0)$.

The subgroup of $\pi_1(X, x_0)$ which operates trivially on $\pi_1(X, x_0)$ itself is precisely the center of $\pi_1(X)$, hereafter denoted by $Z(\pi_1(X))$. Thus $P(X, x_0) \subseteq Z(\pi_1(x))$ so we have $G(X) \subseteq Z(\pi_1(X))$.

COROLLARY I. 5. *If T is any 1-dimensional polyhedron except for the homotopy circle, then $G(T) = 1$.*

COROLLARY I. 6. *Let P^n be the projective space of dimension n. Then $G(P^{2n}) = 1$ for $n > 0$.*

Proof. P^{2n} is not a $2n$-simple space as is well known. That is $\pi_1(P^{2n})$ does not act trivially on $\pi_{2n}(P^{2n})$. Since $\pi_1(P^{2n}) \cong Z_2$, this means that the generator, α, of $\pi_1(P^{2n})$ does not act trivially on $\pi_{2n}(P^{2n})$. Hence $\alpha \notin P(P^{2n}, x_0)$, so $P(X, x_0)$ is the trivial subgroup. Thus $G(P^{2n})$ is trivial.

COROLLARY I. 7. *If M is any closed 2-dimensional manifold with the exception of the torus and the Klein, then $G(M) = 1$.*

Proof. If $M = P^2$, $G(M) = 1$ by the preceding corollary. Otherwise $\pi_1(M)$ has a trivial center as is well known. Hence $G(M) = 1$.

For two of the exceptional cases to Corollaries 5 and 7, the circle S^1 and the torus T, we see that $\pi_1(S^1) = G(S^1)$ and $\pi_1(T) = G(T)$. This result follows from the fact S^1 and T are both topological groups and the following theorem.

THEOREM I. 8. *If X is an H-space, then $G(X)=\pi_1(X)$.*

Proof. An H-space (X,l) has a continuous multiplication and an element l such that right and left multiplication are both homotopic to the identity on X. Since we are assuming that X is a $C.W.$ complex, $X \vee X$ is a subcomplex of $X \times X$ and so has the homotopy extension property. Hence there exists a continuous multiplication, $\cdot$, such that l is a multiplicative identity.

Let $\sigma\colon I \to X$ be any closed path in X such that $\sigma(0)=\sigma(1)=l$.

Define a cyclic homotopy as follows; $h_t(x)=\sigma(t)\cdot x$. The trace $\tau(t)=h_t(l)=\sigma(t)\cdot l=\sigma(t)$. Thus every closed loop at the identity is the trace of some cyclic homotopy, hence $G(X)=\pi_1(X)$.

§ 3. Properties of $G(X,x_0)$.

Any map $f\colon (X,x_0)\to(Y,y_0)$ induces a homomorphism $f_*\colon \pi_1(X,x_0) \to \pi_1(Y,y_0)$. This important property is not enjoyed by $G(X,x_0)$. That is $f_*(G(X))$ is not necessarily contained in $G(Y)$. This may be seen as follows.

Let Y be the figure eight. Let $i\colon (S^1,s_0)\to(Y,y_0)$ be the embedding of the circle onto one of the loops of Y. Now let α be a generator of $\pi_1(S^1,s_0)$. Then $i_*(\alpha)$ is not equal to the identity 1, of $\pi_1(Y,y_0)$. Since $G(Y,y_0)=1$, $i_*(\alpha)\notin G(Y,y_0)$. On the other hand, since $G(S^1)=\pi_1(S^1)$, $\alpha \in G(S^1,s_0)$. Thus $i_*(G(S^1)) \not\subset G(Y)$.

All is not lost, for we do get the following theorems.

THEOREM I. 9. *Let $r\colon (X,x_0)\to(Y,y_0)$ be a retraction. Then*

$$r_*(G(X,x_0)) \subseteq G(Y,y_0).$$

Proof. Let $i\colon Y\to X$ be the inclusion map. Let $y_0\in Y$ be the base point of Y and $i(y_0)$ be the base point of X. Let $[\alpha]\in G(X,i(y_0))$. Then there exists a map $K\colon X\times S^1\to X$ such that $K \mid X = 1_X$ and $K \mid S^1=\alpha$.

Define a map $H\colon Y\times S^1\to Y$ by setting $H(y,s)=r\circ K(i(y),s)$ for $y\in Y$ and $s\in S^1$. Now $H(y,s_0)=r\circ K(i(y),s_0)=r(i(y))=y$ and $H(y_0,s)=r\circ K(i(y_0),s)=r(\alpha(s)) \doteq r\circ\alpha(s)$. Hence $[r\circ\alpha]\in G(Y,y_0)$. But $r_*[\alpha]=[r\circ\alpha]$, so $r_*(G(X,i(y_0)) \subseteq G(Y,y_0)$.

Now consider any x_0 such that $r(x_0)=y_0$. Let σ be a path such that $\sigma(0)=i(y_0)$ and $\sigma(1)=x_0$. Then σ induces an isomorphism $\sigma_*\colon \pi_1(X,x_0) \cong \pi_1(X,i(y_0))$ as follows

$$\sigma_*[\alpha]=[\sigma\cdot\alpha\cdot\sigma^{-1}].$$

Let $[\alpha] \in G(X, x_0)$. By Theorem I.3, $\sigma_*[\alpha] \in G(X, i(y_0))$ and so $r_*(\sigma_*[\alpha]) \in G(Y, y_0)$. But $r_*(\sigma_*[\alpha]) = r_*[\sigma \cdot \alpha \cdot \sigma^{-1}] = [r \circ \sigma \cdot r \circ \alpha \cdot r \circ \alpha^{-1}]$. Since $r \circ \sigma$ and $r \circ \sigma^{-1}$ are closed paths in Y, we have

$$r_*(\sigma_*[\alpha]) = [r \circ \sigma] \cdot [r \circ \alpha] \cdot [r \circ \sigma^{-1}] = [r \circ \sigma] \cdot [r \circ \sigma] \cdot [r \circ \sigma]^{-1}.$$

Since $r_*(\sigma_*[\alpha]) \in G(Y, y_0) \subseteq Z(\pi_1(Y, y_0))$, we have

$$r_*(\sigma_*[\alpha]) = [r \circ \alpha]^{-1} \cdot r_*(\sigma_*[\alpha]) \cdot [r \circ \alpha]$$
$$= [r \circ \alpha]^{-1} \cdot ([r \circ \sigma] \cdot [r \circ \alpha] \cdot [r \circ \sigma]^{-1}) \cdot [r \circ \sigma] = [r \circ \alpha].$$

Therefore $r_*[\alpha] \in G(Y, y_0)$.

If $f: (X, x_0) \to (Y, y_0)$ is a homotopy equivalence, then f induces an isomorphism $f_*: G(X, x_0) \to G(Y, y_0)$. This results from the following theorem.

THEOREM I.10. *If $f: (X, x_0) \to (Y, y_0)$ is a homotopy equivalence, then $f_*(G(X, x_0)) = G(Y, y_0)$.*

Proof. That $f: X \to Y$ is a homotopy equivalence implies the existence, by definition, of a map $g: Y \to X$ such that $f \circ g \cong 1_Y$ and $g \circ f \cong 1_X$. Since y_0 has the Homotopy Extension Property in Y, we may assume that $g(y_0) = x_0$. Let $J: Y \times I \to Y$ be a homotopy such that $J(y, 0) = f \circ g(y)$ and $J(y, 1) = y$.

Now let $[\sigma] \in G(X, x_0)$. Since $f_*: \pi_1(X, x_0) \cong \pi_1(Y, y_0)$ is an isomorphism, we need merely to show that $f_*[\sigma] = [f \circ \sigma] \in G(Y, y_0)$. Let $h_t: X \to X$ be a cyclic homotopy such that $h_t(x_0) = \sigma(t)$ for all $t \in I$. Define a homotopy

$$K: Y \times I \to Y \text{ by } K(y, t) = (f \circ h_t \circ g)(y).$$

Then $k(y, 0) = f \circ g(y) = K(y, 1)$ and

$$k(y_0, t) = f(h_t(x_0)) = f(\sigma(t)) = f \circ \sigma(t)$$

for all $t \in I$.

Define $T: Y \times I \to Y$ such that

$$\begin{aligned} T(y, t) &= J(y, 1 - 3t) \text{ for } 0 \leqq t \leqq \tfrac{1}{3} \\ T(y, 3t - 1) &= K(y, 3t - 1) \text{ for } \tfrac{1}{3} \leqq t \leqq \tfrac{2}{3} \\ T(y, 3t - 2) &= J(y, 3t - 2) \text{ for } \tfrac{2}{3} \leqq t \leqq 1. \end{aligned}$$

Now $T(y, 0) = J(y, 1) = y$ and $T(y, 1) = J(y, 1) = y$, so T is a cyclic homotopy.

Let $\alpha: I \to Y$ be the path given by $\alpha(t) = J(y_0, t)$. Now $\alpha(0) = J(y_0, 0) = f \circ g(y_0) = f(x_0) = y_0$, and $\alpha(1) = J(y_0, 1) = y_0$, so α is a closed path. So the trace of T at y_0, τ, is given by:

$$\tau(t) = T(y_0, t) = (\alpha^{-1} \cdot (f \circ \sigma) \cdot \alpha)(t).$$

Hence $[\tau] = [\alpha]^{-1} \cdot [f \circ \sigma] \cdot [\alpha] \in G(Y, y_0)$. Hence $[f \circ \sigma] \in G(Y, y_0)$ since $G(Y) \subseteq Z(\pi_1(Y))$.

Another property G shares in common with the fundamental group is the following.

THEOREM I. 11. $G(X \times Y, (x_0, y_0)) \cong G(X, x_0) \oplus G(Y, y_0)$.

Proof. Let $Z = X \times Y$ and $Z_0 = (X, y_0)$. There exists an isomorphism

$$h: \pi_1(Z, Z_0) \to \pi_1(X, y_0) \oplus \pi_1(Y, y_0),$$

such that

$$h([\alpha]) = p_*([\alpha]) \oplus q_*([\alpha])$$

where p_* and q_* are induced homomorphisms from the projections of Z onto X and Y respectively. Now $h(G(Z)) \subseteq G(X, x_0) \oplus G(Y, y_0)$ as may readily be seen by noting that projections are retractions and applying Theorem I. 9 to the definition of h.

On the other hand, let $[\alpha]$ and $[\beta]$ be elements of $G(X, x_0)$ and $G(Y, y_0)$ respectively.

Now $h^{-1}([\alpha] \oplus [\beta]) = [(j \circ \alpha) \cdot (k \circ \beta)]$ where j and k inject $X \to X \times y_0$ and $Y \to x_0 \times Y$ respectively.

Since $[\alpha]$ and $[\beta]$ are in $G(X, x_0)$ and $G(Y, y_0)$ respectively, there exists cyclic homotopies H and J having traces α and β respectively.

Let $K: X \times Y \times I \to X \times Y$ be defined as follows:

$$K(x, y, t) = (H(x, 2t), y) \text{ for } 0 \leqq t \leqq \tfrac{1}{2}$$
$$K(x, y, t) = (x, J(y, 2 - 2t)) \text{ for } \tfrac{1}{2} \leqq t \leqq 1.$$

It can easily be verified that K is a cyclic homotopy on $X \times Y$ with trace $(j \circ \alpha) \cdot (k \circ \beta)$. Hence $h^{-1}([\alpha] \oplus [\beta]) \in G(Z, z_0)$, so $h^{-1}(G(X) \oplus G(Y)) \subseteq G(Z)$. Hence $h(G(Z)) \supseteq G(X) \oplus G(Y)$.

§ 4. Aspherical Spaces.

The fact that $G(X) \subseteq P(X)$ leads naturally to the questions; Is there a space X for which $G(X) \neq P(X)$, and if so, under what conditions does equality obtain? The author has not been able to answer these questions, but they have stimulated the next important theorem.

For every $[\sigma] \in \pi_1(X, x_0)$, we can define $f_\sigma: X \vee S^1 \to X$ such that $f_\sigma \mid X = 1_X$ and $f_\sigma \mid S^1 = \sigma$. Let $f_\sigma^{(n+1)}: (X \vee S^1) \cup (X^{(n)} \times S^1) \to X$ be an extension of f_σ where $X^{(n)}$ is the n-skeleton of X. If $f_\sigma^{(n+1)}$ exists, we say that $[\sigma]$ is $(n+1)$-*extensible*. The set of all $(n+1)$-extensible $[\sigma]$ forms a subgroup of $\pi_1(X_1, x_0)$ which we shall denote by $G^{(n)}(X, x_0)$. We get a descending sequence of groups as follows;

$$G^{(1)}(X) \supseteq G^{(2)}(X) \supseteq \cdots \supseteq G(X).$$

On the other hand, let $P^{(n)}(X, x_0)$ stand for the subgroup of $\pi_1(X, x_0)$ of all $[\sigma]$ which operate trivially on $\pi_k(X, x_0)$ for $k \leqq n$. Then

$$P^{(1)}(X) \supseteq P^{(2)}(X) \supseteq \cdots \supseteq P(X).$$

THEOREM I. 12. $G^{(1)}(X, x_0) = P^{(1)}(X, x_0) = Z(\pi_1(X, x_0))$.

Proof. $P^{(1)}(X, x_0) = Z(\pi_1(X, x_0))$ is well known. To prove $G^{(1)}(X, x_0) = Z(\pi_1(X, x_0))$, we must show that $f_\sigma^{(2)}$ exists iff $[\sigma] \in Z(\pi_1(X_1, x_0))$. That is, we must show that $f_\sigma: X \vee S^1 \to X$ is 2-extensible over $X \times S^1$ iff $[\sigma] \in Z(\pi_1(X, x_0))$. This shall be shown by appealing to the following result, which can be found in [4], p. 194.

Remark III. Let L be a connected subcomplex of K containing v_0 and $f: (L, v_0) \to (Y, y_0)$ be a map into a pathwise connected space Y. Then f and the inclusion map $i: L \subset K$ induce the homeomorphisms

$$f_*: \pi_1(L, v_0) \to \pi_1(Y, y_0), \quad i_*: \pi_1(L, v_0) \to \pi_1(K, v_0).$$

Then f is 2-extensible over K iff there exists a homomorphism $h: \pi_1(K, v_0) \to \pi_1(Y, y_0)$ such that $f_* = hi_*$.

For the case at hand, let $L = X \vee S^1$, $Y = X$, $K = X \times S^1$ and $f = f_\sigma$. Then $\pi_1(L) \cong \pi(X) * \pi(S^1)$, the free product, and $\pi_1(K) \cong \pi_1(X) \oplus \pi_1(S^1)$.

Now let $\alpha \in \pi_1(X)$ and $\beta \in \pi_1(S^1)$. Let multiplication between elements of $\pi_1(X)$ be expressed by $(\cdot)$. Let ν generate $\pi_1(S^1)$. Then $i_*(\alpha * \beta) = \alpha \oplus \beta$ and $f_*(\alpha * \nu) = \alpha \cdot [\sigma]$.

Suppose that h exists such that $f_* = hi_*$. Now $f_*(\nu) = [\sigma]$ and $i_*(\nu) = 1 \oplus \nu$, so $[\sigma] = h(1 \oplus \nu)$. On the other hand, $\alpha = f_*(\alpha) = hi_*(\alpha) = h(\alpha \oplus 1)$.

Thus h, if it exists, must satisfy the equation $h(\alpha \oplus \nu) = \alpha \cdot [\sigma]$ for all $\alpha \in \pi_1(X)$. Now $f_*(\alpha * \nu) = \alpha \cdot [\sigma]$ and $f_*(\nu * \alpha) = [\sigma] \cdot \alpha$. But $hi_*(\alpha * \nu) = \alpha \cdot [\sigma] = hi_*(\nu * \alpha)$. Hence $\alpha \cdot [\sigma] = [\sigma] \cdot \alpha$ for all $\alpha \in \pi_1(X)$.

The above theorem enables us to determine precisely what $G(X)$ is when X is aspherical, i. e., when $\pi_n(X) = 0$ for $n > 1$.

COROLLARY I.13. *If X is aspherical, then $G(X,x_0)=Z(\pi_1(X,x_0))$.*

Proof. Since X is aspherical, $X\times S^1$ is aspherical. Thus any map $f_\sigma\colon X\vee S^1\to X$ which is 2-extensible must be extensible over $X\times S^1$.

The above corollary permits us to settle the one holdout among the closed, 2-dimensional manifolds the Klein Bottle K.

COROLLARY I.14. *Let K be the Klein Bottle. Then $G(K)=Z(\pi_1(K))$.*

II. The universal covering space. As in the first chapter, X will always be a pathwise-connected $C.W.$-complex. This is enough to insure the existence of the universal covering space C. We shall let $p\colon(C,\tilde{x}_0)\to(X,x_0)$ be the covering projection.

§1. *The Universal Covering Space and $G(X,x_0)$.*

There is a natural isomorphism, ν, between $\pi_1(X,x_0)$ and the group of Deck Transformations, $\mathcal{D}(X)$, acting on C. Thus $G(X,x_0)$ corresponds to a subgroup of $\mathcal{D}(X)$ under ν. This subgroup, $\nu G(X)$, has a natural definition within $\mathcal{D}(X)$.

THEOREM II.1. *$G(X,x_0)$ is isomorphic to the subgroup of those Deck Transformations which are homotopic to 1_C by fiber preserving homotopies.*

Proof. Suppose that $[\lambda]\in\pi_1(X,x_0)$ gives rise to the deck-transformation $l\colon C\to C$. This means that any path α from $\tilde{x}_0$ to $l(\tilde{x}_0)$ projects down upon the closed path $p\circ\alpha\in[\lambda]$.

Now suppose that $[\lambda]\in G(X,x_0)$. Then there exists a cyclic homotopy $h_t\colon X\to X$ whose trace is λ. Now $1_C\colon C\to C$ covers the map $1_X\circ p\colon C\to X$. Since $h_t\circ p\colon C\to X$ is a homotopy of $1_X\circ p$, by the Covering Homotopy Property there must exist a homotopy $\tilde{h}_t\colon C\to C$ which lifts $h_t\circ p$. That is $h_t\circ p=p\circ\tilde{h}_t$. Now $h_1=1_X$ so $p=p\circ\tilde{h}_1$. Thus $\tilde{h}_1$ must be a deck transformation of C. Now $\tilde{h}_1=l$ since the path $\tilde{\tau}(t)=\tilde{h}_t(\tilde{x}_0)$ running from $\tilde{x}_0$ to $\tilde{h}_1(\tilde{x}_0)$ lifts λ. So $\tilde{h}_t$ is the required fiber preserving homotopy from 1_C to l.

Conversely, if $\tilde{h}_t$ is a fiber preserving homotopy such that $\tilde{h}_0=1_C$ and $\tilde{h}_1=l$, then there exists a cyclic homotopy $h_t\colon X\to X$ such that $h_t\circ p=p\circ\tilde{h}_t$. Clearly h_t is a cyclic homotopy and its trace $\tau(t)=h_t(x_0)$ is contained in $[\lambda]$.

For covering spaces, fiber-preserving homotopies satisfy a very nice condition.

THEOREM II.2. *The homotopy $\tilde{h}_t\colon C\to C$ is fiber preserving iff $f\circ\tilde{h}_t=\tilde{h}_t\circ f$ for every $f\in\mathcal{D}(X)$.*

Proof. Suppose $\tilde{h}_t$ is fiber preserving. Let f be any deck transformation and $x \in C$ any point. Then $f(x)$ and x are in the same fiber. Since $\tilde{h}_t(x)$ and $\tilde{h}_t(f(x))$ are both in the same fiber, there is a $g \in \mathfrak{D}(X)$ such that $g \circ \tilde{h}_t(x) = \tilde{h}_t \circ f(x)$. If $\epsilon > 0$ is sufficiently small, $g \circ \tilde{h}_{t-\epsilon}(x) = \tilde{h}_{t-\epsilon} \circ f(x)$. Thus the greatest lower bound of the set of t's such that $g \circ \tilde{h}_t(x) = \tilde{h}_t \circ f(x)$ must occur when $t = 0$. Therefore by continuity, $g \circ \tilde{h}_0(x) = \tilde{h}_0 \circ f(x)$. But $\tilde{h}_0 = 1_C$, so $g(x) = f(x)$. This can occur only when $g = f$. Thus $f \circ \tilde{h}_t = \tilde{h}_t \circ f$ for all $f \in \mathfrak{D}(X)$.

On the other hand, suppose $f \circ \tilde{h}_t = h_t \circ f$ for all $f \in \mathfrak{D}(X)$. Let x and y both be in the same fiber of p and suppose that $f(x) = y$. Now $\tilde{h}_t = f^{-1} \circ \tilde{h}_t \circ f$, so $\tilde{h}_t(x) = f^{-1} \circ \tilde{h}_t(y)$. Thus $\tilde{h}_t(x)$ is in the same fiber as $\tilde{h}_t(y)$. Hence $\tilde{h}_t$ is fiber preserving.

COROLLARY II. 3. *$G(X, x_0)$ is isomorphic to the subgroup of $\mathfrak{D}(X)$ given by those deck transformations which are homotopic to the identity by a homotopy which commutes with every deck transformation.*

§ 2. Computations.

Let p and q be relatively prime integers. Then $L(p, q)$, a three dimensional lens space, has a fundamental group isomorphic to the cyclic group of order P.

THEOREM II. 4. $G(L(p,q)) = \pi_1(L(p,q))$.

Proof. Let S^3 be the 3-sphere given by the complex coordinates (Z_0, Z_1) such that $Z_0\bar{Z}_0 + Z_1\bar{Z}_1 = 1$. Then let $f\colon S^3 \to S^3$ such that

$$f(Z_0, Z_1) = (Z_0 e^{2\pi i/p}, Z_1 e^{2\pi q i/p}).$$

Now f generates a cyclic group Z_p of rotations, each element of which is fixed point free. The factor space S^3/Z_p is the lens space $L(p,q)$. [3, page 262]

Now let $h_t\colon S^3 \to S^3$ such that $h_t(Z_0, Z_1) = (Z_0 e^{2\pi t i/p}, Z_1 e^{2\pi q t i/p})$ be a homotopy. Now h_0 is the identity on S^3 and $h_1 = f$. Also $f \circ h_t = h_t \circ f$, so h_t commutes with all Z_p. Therefore $f \in \nu G(L(p,q))$, hence $G(L(p,q)) \cong Z_p$.

THEOREM II. 5. *Let P^n be the real projective space of dimension n. Then $G(P^{2n+1}) = \pi_1(P^{2n+1}) \cong Z_2$.*

Proof. S^{2n+1} can be given by the $n+1$ tuple of complex numbers $(Z_0, \cdots, Z_n)$ satisfying the equation $\bar{Z}_0 Z_0 + \cdots + \bar{Z}_n Z_n = 1$. The projective space P^{2n+1} is created by identifying antipodal points. Let f be the deck transformation such that $f(Z_0, \cdots, Z_n) = (-Z_0, \cdots, -Z_n)$. Define a homotopy $h_t\colon S^{2n+1} \to S^{2n+1}$ such that $h_t(Z_0, \cdots, Z_n) = Z_0 e^{\pi t i}, \cdots, Z_n e^{\pi t i})$.

Then h_0 is the identity and $h_1 = f$. Also $f \circ h_t = h_t \circ f$. So by Corollary II. 3, $G(P^{2n+1}) = \pi_1(P^{2n+1}) \cong Z_2$.

§ 3. $\mathfrak{H}(X)$.

Definition. Let $\mathfrak{H}(X)$ be the set of all those deck transformations in the center of $\mathfrak{D}(X)$ which are homotopic to the identity. It is easy to verify that $\mathfrak{H}(X)$ is a subgroup of $\mathfrak{D}(X)$. We shall use the same symbol, $\mathfrak{H}(X)$, to stand for the corresponding subgroup in $\pi_1(X, x_0)$.

THEOREM II. 6. $G(X, x_0) \subseteq \mathfrak{H}(X) \subseteq P(X, x_0)$.

Proof. That $G(X, x_0) \subseteq \mathfrak{H}(X)$ is obvious from Corollary II. 3.

Let $f \in \mathfrak{D}(X)$ such that $f \cong 1_X$. Let $h_t \colon C \to C$ be the homotopy such that $h_0 = 1_X$ and $h_1 = f$. Let $\tilde{\phi} \colon I \to C$ such that $\tilde{\phi}(t) = h_t(x_0)$. Let $\phi = p \circ \tilde{\phi}$. Then f corresponds to $[\phi]$ under ν.

Suppose ϕ operates on $[\alpha] \in \pi_n(X, x_0)$, $n > 1$. Then ϕ operates trivially on α iff the map $g \colon S^n \vee S^1 \to X$ such that $g \mid S^n = \alpha$ and $g \mid S^1 = \phi$ can be extended to a map $g' \colon S^n \times S^1 \to X$.

We define g' as follows. There exists a map $l \colon S^n \to C$ such that $p \circ l = g \mid S^n$ for $n > 1$. This is a well known property of the universal covering space. If we consider S^1 as the unit interval such that 0 and 1 are identified, then $g'(s, t) = p \circ h_t \circ l, s)$. Since $g'(s, 0) = P(l(s)) = p \circ f(l(s)) = p \circ h_1 \circ l(s) = g'(s, 1)$, g' is well defined and it is easily verified that g' is an extension of g. So we have shown that $\mathfrak{H}(X)$ is contained in the subgroup of all elements of $\pi_1(X, x_0)$ which operates trivially on $\pi_n(X, x_0)$ for $n > 1$. Since $\mathfrak{H}(X) \subseteq Z(\pi_1(X, x_0))$, $\phi \in \mathfrak{H}(X)$ operates trivially on $\pi_1(X, x_0)$, hence $\mathfrak{H}(X) \subseteq P(X, x_0)$.

We have no examples which indicate whether the inclusions are proper. Certainly, for spaces whose universal covering spaces are compact, odd-dimensional homotopy spheres, $\mathfrak{H}(X) = P(X) = Z(\pi_1(X))$. In particular, if X is a compact three-dimensional manifold with finite fundamental group, then $\mathfrak{H}(X) = Z(\pi_1(X))$.

§ 4. Aspherical Spaces.

THEOREM II. 7. *Let X be a contractible C. W.-complex with π, a discrete group of homeomorphisms of X onto itself, acting freely on X. Then if $f \in Z(\pi)$, there is a homotopy h_t which commutes with g for all $g \in \pi$ such that $h_0 = 1_X$ and $h_1 = f$.*

Proof. If we let X/π stand for the space obtained by identifying the orbits under π, then X may be regarded as the universal covering space of X/π. Thus π may be regarded as the deck transformations of the covering and hence also as the fundamental group of X/π. Since X/π is aspherical, $G(X/\pi) \cong Z(\pi)$. Hence the center of π consists of homeomorphisms of X which are homotopic to the identity by a homotopy h_t such that $f \circ h_t = h_t \circ f$.

III. X^X.

Let X^X denote the space of continuous mappings from X into X with the compact-open topology. Let Ω be the path connected component of X^X which contains the identity 1_X.

Let $p: X^X \to X$ be the evaluation $p(f) = f(x_0)$. Since we wish p to be continuous, we will assume that X is locally compact throughout this chapter. We also avoid complications if p is a fibering, and this occurs when X is a locally finite simplicial polyhedron. With the help of p, we can characterize $G(X, x_0)$.

Remark IV. There is a natural homeomorphism between the space of maps $(X^X)^{S^n}$ and $X^{X \times S^n}$ given by $\phi: (X^X)^{S^n} \to X^{X \times S^n}$ such that $\phi(f)(x, s) = (f(s))(x)$ for $x \in X$ and $s \in S^n$. Note that $f \cong g$ iff $\phi(f) \cong \phi(g)$.

THEOREM III. 1. $p_* \pi_1(X^X, 1_X) = G(X, x_0)$.

Proof. By the remark, the closed path $f: S^1 \to X^X$ corresponds to the cyclic homotopy $\phi(f): X \times S^1 \to X$. Now $p \circ f: S^1 \to X$ is equal to $\phi(f) \mid S^1$ for $p(f)(x_0, s) = f(s)(x_0) = p(f(s)) = p \circ f(s)$.

This is to say that every closed loop f in $\Omega \subseteq X^X$ is a cyclic homotopy of X whose trace equals $p \circ f$ and conversely, every cyclic homotopy of X is a closed path f in Ω such that $p \circ f$ equals the trace of the cyclic homotopy.

THEOREM III. 2. *Let X be a locally finite, aspherical, pathwise connected simplicial polyhedron. Then $p_*: \pi_1(X^X, 1_X) \cong Z(\pi_1(X, x_0))$ and $\pi_n(X^X, 1_X) = 0$ for $n > 1$.*

Proof.

LEMMA 1. $\pi_n(X^X, 1_X) = 0$ *if* $n > 1$.

Proof. Let $f: X \times S^n \to X$ such that $f(X, s_0) = x$ for all $x \in X$ where $s_0 \in S^n$ is the base point of S^n. Define $d: X \times S^n \to X$ such that $d(x, s) = x$, that is d is the projection of $X \times S^1$ onto X. By the remark, if we can show that $f \cong d$, then $\phi^{-1}(f) \cong \phi^{-1}(d)$. Since $\phi^{-1}(d): S^1 \to X^X$ is the constant map onto 1_X, this will prove the lemma.

Since X is aspherical, $f \cong d$ iff $f_*\colon \pi_1(X \times S^n, x_0 \times s_0) \to \pi_1(X, x_0)$ and $d_*\colon \pi_1(X \times S^n, x_0 \times s_0) \to \pi_1(X, x_0)$ are equivalent, that is

$$f_*(\alpha) = \xi^{-1} \cdot g_*(\alpha) \cdot \xi$$

for all $\alpha \in \pi_1(X \times S^n)$ and some $\xi \in \pi_1(X)$. See Hu [4], pp. 198-9.

Now $\pi_1(X \times S^n, x_0 \times s_0) \cong \pi_1(X, x_0) \oplus \pi_1(S^n, s_0) \cong \pi_1(X, x_0)$. Both f_* and d_* "act like the identity" and so $f_* = d_*$. Hence $f \cong d$.

LEMMA 2. $p_*(\pi_1(X^X, 1_X)) = Z(X, x_0))$.

Proof. Since X is aspherical, $Z(\pi_1(X, x_0)) = G(X, x_0)$. Hence by the preceding theorem, the lemma is true.

LEMMA 3. *Let $\Omega_0 \subseteq X^X$ be the space of maps such that $f(x_0) = x_0$ for all $f \in \Omega_0$. Then $\pi_1(\Omega_0, 1_X) = 0$.*

Proof. Let $d\colon X \times S^1 \to X$ such that $d(x, s) = x$. Let f be any arbitrary $f\colon X \times S^1 \to X$ such that $f(x_0, s) = x_0$ for all $s \in S^1$. We will prove the lemma by showing there is a homotopy $h_t\colon X \times S^1 \to X$ such that $h_0 = f$ and $h_1 = d$ and $h_t(x_0, s) = x_0$ for all $t \in I$ and $s \in S^1$. For then $\phi^{-1}(h_t)$ will be a homotopy connecting $\phi^{-1}(f) \in \Omega_0$ and $\phi^{-1}(d)$ which is the constant map $S^1 \to 1_X$. Since $\phi^{-1}(h_t) \in \Omega_0$ for each $t \in I$, the lemma will be proved.

We may regard S^1 as I with the points 0 and 1 identified. Thus we may regard f and d as maps from $X \times I$ into X.

Let

$$A = (X \times 0 \times I) \cup (X \times 1 \times I) \cup (x_0 \times I \times I) \cup (X \times I \times 0) \cup (X \times I \times 1).$$

Define $H^{(1)}\colon A \to X$ such that

$$\begin{aligned} H^{(1)}(x, s, 0) &= f(x, s) \\ H^{(1)}(x, s, 1) &= d(x, s) \\ H^{(1)}(x_0, s, t) &= x_0 \\ H^{(1)}(x, 0, t) &= H(x, 1, t) = x. \end{aligned}$$

We wish to extend $H^{(1)}$ to a map $H\colon X \times I \times I \to X$. Then $H(x, s, t) = h_t(x, s)$ will give us the homotopy mentioned above, which will prove the lemma.

Let $X^{(n)}$ be the n-skeleton of X. Let $K = X \times I \times I$. Regard I as being decomposed into $\{0\}$, $\{1\}$ and $(0, 1)$. Then

$$K^{(1)} \subseteq A \text{ and } K^{(2)} \subseteq X^{(0)} \times I \times I \cup A.$$

We shall extend $H^{(1)}: A \to X$ to $H^{(2)}: K^{(2)} \to X$ by the following procedure. Let $x_i \in X^{(0)}$. Then

$$S_i^1 = (x_i \times I \times 0) \cup (x_i \times 1 \times I) \cup (x_i \times I \times 1) \cup (x_i \times 0 \times I)$$

forms a circle. Since $S_i^1 \in A$, $H^{(1)} \mid S_i^1 : S_i^1 \to X$. It is easily seen, that $H^{(1)} \mid S_i^1$ is null homotopic and hence may be extended to $H_i^{(2)}: X_i \times I \times I$.

Define $H^{(2)}: K^{(2)} \to X$ by

$$H^{(2)}(y) = H^{(1)}(y) \text{ if } y \in A$$
$$H^{(2)}(y) = H_i^{(2)}(y) \text{ if } y \in x_i \times I \times I.$$

Since X is aspherical, we may extend $H^{(2)}: K^{(2)} \to X$ to $H: X \times I \times I \to X$. Since $H(x, 0, t) = H(x, 1, t)$, H may be regarded as a map from $X \times S^1 \times I$ to X. Now we can define $h_t(x, s) = H(x, s, t)$ and we see that $h_0 = f$ and $h_1 = d$ and $h_t(x_0, s) = x_0$.

LEMMA 4. $p_*: \pi_1(X^X, 1_X) \cong Z(\pi_1(X, x_0))$.

Proof. Consider the homotopy sequence

$$\pi_1(\Omega_0) \xrightarrow{i_*} \pi_1(X^X) \xrightarrow{p_*} \pi_1(X).$$

Since $\pi_1(\Omega_0) = 0$, p_* must be 1-1. But $p_*\pi_1(X^X) = Z(\pi_1(X))$.

Lemmas 1 through 4 prove the theorem.

COROLLARY III. 3. *If X is a pathwise connected aspherical locally finite simplicial polyhedron, then Ω, the path component of X^X containing 1_X, is contractible when $Z(\pi_1(x)) = 1$.*

Proof. By Milnor [6], Ω has the homotopy type of a *C. W.*-complex. Since $\pi_n(\Omega) = 0$ for all n, by a theorem of Whitehead's [8], Ω is contractible.

COROLLARY III. 4. *If X is a pathwise connected, aspherical, locally finite simplicial polyhedron, then $p: \Omega \to X$ is a homotopy equivalence iff $\pi_1(X, x_0)$ is abelian.*

Proof. Again by Milnor [6] and Whitehead [8].

IV. The Euler-Poincaré number and $G(X)$.

THEOREM IV. 1. *Suppose X has the same homotopy type as a compact, connected polyhedron. Then if the Euler-Poincaré number $\chi(X)$ is not equal to zero, $G(X)$ is trivial.*

Proof. By Theorem I. 10, we may assume that X is a compact, connected polyhedron.

The proof is a simple application of the Nielsen-Wecken theory of fixed point classes. We shall summarize the pertinent facts needed for the proof. These are proved in Wecken [7] and are in the notation of Jaing Bo-Ju in [1].

Let $\tilde{X}$ be the universal covering of X. We regard $\pi_1(X)$ as the group of deck transformations on $\tilde{X}$. Let $f: X \to X$. Consider the set of all lifts of f to maps $\tilde{f}: \tilde{X} \to \tilde{X}$. We define an equivalence relation among these lifts as follows: $\tilde{f} \equiv \tilde{f}_1$ if and only if $\tilde{f}_1 = \gamma^{-1} \circ \tilde{f} \circ \gamma$ for some $\gamma \in \pi_1(X)$. Let $[\tilde{f}]$ denote the equivalence class of $\tilde{f}$. The set of fixed points of $\tilde{f}$ project down, by the covering map p, onto a subset of fixed points of $\tilde{f}$. The fixed points of any $\tilde{f}_1$ in the same equivalence class as $\tilde{f}$ also project down to the same subset of fixed points of f. If $\tilde{f}_1$ is not equivalent to f, then the fixed points of $\tilde{f}_1$ project down to a subset of fixed points of f disjoint from those of $\tilde{f}$. This procedure partitions the fixed points of f into disjoint subsets, called fixed point classes. Thus each fixed point class is uniquely associated with an equivalence class of lifts of f. We can also have lifts, $\tilde{f}$, of f with no fixed points, and so the equivalence class of $\tilde{f}$ corresponds to a void class of fixed points.

If $h_t: f \cong g$ for $g: X \to X$, then h_t defines a 1-1 correspondence between the lifts of f and those of g preserving equivalence classes. Hence there is a 1-1 correspondence between fixed point classes.

With each fixed point class $[\tilde{f}]$, it is possible to assign a number ν such that $\nu = 0$ if $[\tilde{f}]$ is empty and such that ν is preserved under homotopy. That is if $[\tilde{f}]$ corresponds to $[\tilde{g}]$ under a homotopy from f to g, then ν for $[\tilde{g}]$ is equal to the ν for $[\tilde{f}]$. Finally the sum of all the ν's equals Λ_f, the Lefschitz number.

Suppose that $f = 1_X$. Then every $\nu = 0$ except possibly for ν_1, the number associated with the fixed point class given by the identity $\tilde{1}: \tilde{X} \to \tilde{X}$. This follows since every other lift of 1_X has no fixed point. Also we know that $\Lambda_f = \chi(X)$ when $f = 1_X$. Assume that $\chi(X) \neq 0$. Then $\nu_1 = \chi(X) \neq 0$.

Let $\alpha \in G(X)$. Then there is a cyclic homotopy $h_t: X \to X$ which can be lifted to a homotopy $\tilde{h}_t: \tilde{1} \cong \alpha$ where we regard α as a deck transformation. So $[\tilde{1}]$ corresponds to $[\alpha]$. But $\alpha: \tilde{X} \to \tilde{X}$ has no fixed points, unless $\alpha = \tilde{1}$. Since $\nu \neq 0$ for $[\alpha]$, the associated fixed point class must be non-empty so $\alpha = \tilde{1}$. Thus $\alpha = 1 \in \pi_1(X)$. Hence $G(X) = 1$.

This theorem yields a number of very interesting corollaries.

COROLLARY IV. 2. *Let X be the homotopy type of a connected, compact polyhedron. If X is an H-space and $\chi(X) \neq 0$, then $\pi_1(X) = 1$.*

Proof. By Theorem I. 8, $G(X) = \pi_1(X)$. Hence, since $G(X) = 1$, we have $\pi_1(X) = 1$.

As a matter of fact, it can be shown, using homological properties of H-spaces, that $\chi(X) = 0$ or $\chi(X) = 1$, in which case X is contractible. See [2] for a proof of this in the case of semigroups.

COROLLARY IV. 3. *Let X have the same homotopy type as a connected, compact polyhedron. If $\chi(X) \neq 0$ and X is aspherical, then $Z(\pi_1(X)) = 1$.*

Proof. By Corollary I. 13, $G(X) = Z(\pi_1(X))$. Hence $Z(\pi_1(X)) = 0$.

As an application of this result, we can get the following well known result.

COROLLARY IV. 4. *For any closed 2-dimensional manifold, excepting the torus, projective space and the Klein Bottle, the center of the fundamental group is trivial.*

Corollary IV. 3 also has applications to the imbedding of complexes in spheres. The author is indebted to L. P. Neuwirth for suggesting this line of approach.

COROLLARY IV. 5. *Let X be an m-dimensional, connected subcomplex of S^n where $m \leqq n-2$. Then $S^n - X$ aspherical implies that $Z(\pi_1(S^n - X)) = 1$ provided that $\chi(X) \neq 0$ if n is odd and $\chi(X) \neq 2$ if n is even.*

Proof. Let $\pi^i(Y)$ stand for the i-th Betti number of any topological space Y for $i > 0$. For $i = 0$, $\pi^i(Y)$ will equal the number of connected components of Y minus 1. Now $\pi^p(X) = \pi^{n-p-1}(S^n - X)$ for $0 \leqq p \leqq n-1$ by Alexander's Duality. Then

$$\begin{aligned}\chi(X) &= \sum_{p=0}^{m} (-1)^p \pi^p(X) + 1 \\ &= \sum_{p=0}^{m} (-1)^p \pi^{n-p-1}(S^n - X) + 1 \\ &= \sum_{j=n-1}^{n-m-1} (-1)^{n-j-1} \pi^j(S^n - X) + 1 \\ &= \sum_{j=0}^{n-1} (-1)^{n-j-1} \pi^j(S^n - X) + 1\end{aligned}$$

since $\pi^j(S^n - X) = \pi^{n-j-1}(X) = 0$ for $j < n-m-1$. Also since $\pi^n(S^n - X) = 0$, we have

$$\chi(X) = \sum_{j=0}^{n} (-1)^{n-j-1}\pi^j(S^n - X) + 1$$
$$= (-1)^{n-1}[\sum_{j=0}^{n} (-1)^j\pi^j(S^n - X)] + 1$$

$$\chi(X) = (-1)^{n-1}[\chi(S^n - X) - 1] + 1$$
$$= -(-1)^n\chi(S^n - X) + (-1)^n + 1.$$

Hence we have that

$$\chi(S^n - X) = (-1)^{n-1}\chi(X) + (1 + (-1)^n).$$

So $\chi(S_n - X) \neq 0$, if $\chi(X) \neq 0$ when n is odd and also if $\chi(X) \neq 2$ if n is even.

Now $S^n - X$ is connected and is of the same homotopy type as a closed subcomplex of S^n. Hence apply Corollary IV. 3.

A natural generalization of Theorem IV. 1 is the following: If X is a compact polyhedron and $\chi(X) \neq 0$, then $p_*\pi_n(X^X, 1_X) = 0$ for all n.

This statement is untrue. It is known that the homeotopy sequence [5] gives rise to isomorphisms $p_*: \pi_n(G) \cong \pi_n(S^2)$, $n > 2$ where G is the group of homeomorphisms of S^2 onto itself and p_* is induced by the evaluation map. Since $p_*\pi_n(X^X, 1_X) \supset p_*\pi_n(G, 1_X)$, we see that $p_*\pi_3(X^X, 1_X) = \pi_3(S^2) \cong Z$ if $X = S^2$. $\chi(S^2) \neq 0$ so the above generalization is false.

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