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Correction to “On fibre spaces and the evaluation map”

By DANIEL H. GOTTLIEB

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Guy Allaud has pointed out a gap in the proof of Lemma 2 in [1, § 3]. This lemma is basic to the results of the paper. We shall fill the gap with Lemma 1 which is a strong form of the fact that equivalent fibrations have homotopic classifying maps.

Let $p: E \rightarrow B$ and $p': E' \rightarrow B'$ be fibrations. Suppose we have maps f and \tilde{f} such that the following diagram is commutative:

$$\begin{array}{ccc} E & \xrightarrow{\tilde{f}} & E' \\ \downarrow p & & \downarrow p' \\ B & \xrightarrow{f} & B' \end{array}$$

Then we say that \tilde{f} covers f . Now \tilde{f} restricted to any fibre F in E maps F to a fibre in E' . If \tilde{f} maps the fibres of E to the fibres of E' by homotopy equivalences, then we say that \tilde{f} properly covers f .

LEMMA 1. *Suppose that $p_\infty: E_\infty \rightarrow B_\infty$ is a universal fibration for fibre F and suppose that $p: E \rightarrow B$ is a fibration with fibre F . Suppose that $f: B \rightarrow B_\infty$ and $g: B \rightarrow B_\infty$ are classifying maps for $p: E \rightarrow B$. Let $\tilde{f}, \tilde{g}: E \rightarrow E_\infty$ properly cover f and g respectively. Then \tilde{f} and \tilde{g} are homotopic by a fibre preserving homotopy, written $\tilde{f} \simeq \tilde{g}$.*

It will be convenient to consider an altered form of the above lemma.

LEMMA 2. *There is a map $\tilde{s}: E \rightarrow E_\infty$, which properly covers a classifying map $s: B \rightarrow B_\infty$ for the fibration $p: E \rightarrow B$, having the property that $\tilde{s}h \simeq \tilde{s}$ for any fibre homotopy equivalence $h: E \rightarrow E$.*

Please note that the fibre preserving homotopy between $\tilde{s}h$ and \tilde{s} is not necessarily a fibre-wise homotopy, i.e., the homotopy need not cover s for all values of $t \in I$.

Proof that Lemma 2 \Rightarrow Lemma 1. We shall prove that $\tilde{f} \simeq \tilde{s}$. The argument for $\tilde{g} \simeq \tilde{s}$ is exactly the same, so $\tilde{f} \simeq \tilde{g}$.

Both f and $s: B \rightarrow B_\infty$ are classifying maps for $p: E \rightarrow B$, so f is homotopic to s . By the covering homotopy property, there is a $\tilde{s}_1: E \rightarrow E_\infty$ such that \tilde{s}_1

properly covers f and $\tilde{s}_1 \simeq \tilde{s}$.

Let f^*E_∞ be the total space of the fibration induced by f . Then we obtain the following diagram:

$$\begin{array}{ccccc}
 E & \xrightarrow{\bar{f}} & f^*E_\infty & \xrightarrow{\eta} & E_\infty \\
 \downarrow p & & \downarrow & & \downarrow p_\infty \\
 B & \xrightarrow{1_B} & B & \xrightarrow{f} & B_\infty
 \end{array}$$

Here η is the natural map $f^*E_\infty \rightarrow E_\infty$ and \bar{f} and \bar{s}_1 are the unique maps such that $\eta\bar{f} = \tilde{f}$ and $\eta\bar{s}_1 = \tilde{s}_1$.

Now \bar{f} and \bar{s}_1 both properly cover 1_B , and so, by a result of Dold, \bar{f} and \bar{s}_1 are fibre homotopy equivalences. Let $\bar{f}^{-1}, \bar{s}_1^{-1}: f^*E_\infty \rightarrow E$ be fibre homotopy inverses to \bar{f} and \bar{s}_1 respectively. Now $\bar{s}_1^{-1} \circ \bar{f}$ is a fibre homotopy equivalence from E to E , so by Lemma 2 and the above considerations,

$$\tilde{s} \simeq \tilde{s} \circ (\bar{s}_1^{-1} \circ \bar{f}) \simeq \tilde{s}_1 \circ (\bar{s}_1^{-1} \circ \bar{f}) = \eta \circ \bar{s}_1 \circ \bar{s}_1^{-1} \circ \bar{f} \simeq \eta \circ \bar{f} = \tilde{f}.$$

PROOF OF LEMMA 2. To show the existence of such an \tilde{s} , we will construct a fibration using the techniques of Dold [2, pp. 6.3-6.5]. Let J_1 and J_2 be two closed intervals whose union is the circle and whose intersection consists of two disjoint small intervals. Then $p \times 1: E \times J_\nu \rightarrow B \times J_\nu, \nu = 1, 2$ are fibrations. Here we let $H_\nu = E \times J_\nu$ and $X_\nu = B \times J_\nu$ and $U = B \times (J_1 \cap J_2)$. Then, in the notation of Dold, $H_1^U = H_2^U = E \times (J_1 \cap J_2)$. Any fibre homotopy equivalence $h: E \rightarrow E$ gives rise to a fibre homotopy equivalence $\varphi: H_1^U \rightarrow H_2^U$ by defining $\varphi = h \times 1$ over one component of U and $\varphi = \text{identity}$ over the other component. Using this φ we construct R as in [2, p. 6.4]. Defining $H_1 \cup H_2 \cup R$ as in [2], we obtain a weak fibration, which we shall denote by $E_\varphi \rightarrow B \times S^1$.

We can regard E as a subspace of E_φ under an inclusion map \tilde{i} which takes $e \rightarrow (e, r)$ where $r \in J_1 \cap J_2$. Then we can define a fibre preserving homotopy $h_i: E \rightarrow E_\varphi$ such that $h_0 = \tilde{i}$ and $h_1 = \tilde{i} \circ h$.

Now from each fibre homotopy equivalence class, we select a fibre homotopy equivalence and perform the above constructions. Then we identify the subsets E of each of these E_{φ_i} together. Call the space M . There results a weak fibration $M \rightarrow B \times (\mathbf{V} S^1)$. Since every weak fibration is fibre homotopy equivalent to a Hurewicz fibration, we obtain the following diagram:

$$\begin{array}{ccccccc}
 E & \xrightarrow{\tilde{i}} & M & \xrightarrow{\tilde{j}} & M^* & \xrightarrow{\tilde{k}} & E_\infty \\
 \downarrow p & & \downarrow & & \downarrow & & \downarrow p_\infty \\
 B & \xrightarrow{i} & B \times (\mathbf{V} S^1) & \xleftarrow{1} & B \times (\mathbf{V} S^1) & \xrightarrow{k} & B_\infty
 \end{array}$$

Here M^* is the fibre space and \tilde{j} is a fibre homotopy equivalence and k is the classifying map of M^* and \tilde{k} properly covers k . Then we define our map $\tilde{s} = \tilde{k}\tilde{j}\tilde{i}$. Then any fibre homotopy equivalence $h': E \rightarrow E$ gives rise to a fibre preserving homotopy

$$E \xrightarrow{h'_t} E_{\varphi'} \subset M \xrightarrow{\tilde{j}} M^* \xrightarrow{\tilde{k}} E_{\infty}$$

connecting \tilde{s} with $\tilde{s} \circ h'$.

Lemma 1 \Rightarrow *Lemma 2* of [1]. Let $p: E \rightarrow B$ be a fibration and let $k: B \rightarrow B_{\infty}$ be a classifying map for p . Suppose $\tilde{k}: E \rightarrow E_{\infty}$ properly covers k . Let $i: X \rightarrow CX$ be the inclusion of X into the cone over X . Then we have the commutative diagram

$$\begin{CD} E \times X @>1 \times i>> E \times CX @>\tilde{k} \times \text{const.}>> E_{\infty} \\ @Vp \times 1VV @Vp \times 1VV @Vp_{\infty}VV \\ B \times X @>1 \times i>> B \times CX @>k \times \text{const.}>> B_{\infty} \end{CD}$$

To prove Lemma 2 of [1], we must show that for any map $\tilde{A}: E \times X \rightarrow E_{\infty}$ which properly covers a map $A: B \times X \rightarrow B_{\infty}$, we have $\tilde{A} \simeq (\tilde{k} \times \text{const.}) \circ (1 \times i)$. This follows from Lemma 1 and the fact that both A and $(k \times \text{const.}) \circ (1 \times i)$ are classifying maps of $p \times 1: E \times X \rightarrow B \times X$.

A remark in [1, § 6] should be amended. If $E_* \rightarrow B_*$ is a universal fibre bundle or fibre space for some theory, and if $E_* \rightarrow B_*$ satisfies the proper analogue of Lemma 1, then the analogues of the results of [1] will follow.

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