Correction to "On fibre spaces and the evaluation map"

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These Annals Vol. 87 (1968), 42–55

Guy Allaud has pointed out a gap in the proof of Lemma 2 in [1, § 3]. This lemma is basic to the results of the paper. We shall fill the gap with Lemma 1 which is a strong form of the fact that equivalent fibrations have homotopic classifying maps.

Let $p: E \to B$ and $p': E' \to B'$ be fibrations. Suppose we have maps f and $\tilde{f}$ such that the following diagram is commutative:

$$\begin{array}{ccc} E & \xrightarrow{\tilde{f}} & E' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow[f]{} & B' \end{array}$$

Then we say that $\tilde{f}$ *covers* f. Now $\tilde{f}$ restricted to any fibre F in E maps F to a fibre in E'. If $\tilde{f}$ maps the fibres of E to the fibres of E' by homotopy equivalences, then we say that $\tilde{f}$ *properly covers* f.

LEMMA 1. *Suppose that $p_\infty: E_\infty \to B_\infty$ is a universal fibration for fibre F and suppose that $p: E \to B$ is a fibration with fibre F. Suppose that $f: B \to B_\infty$ and $g: B \to B_\infty$ are classifying maps for $p: E \to B$. Let $\tilde{f}$, $\tilde{g}: E \to E_\infty$ properly cover f and g respectively. Then $\tilde{f}$ and $\tilde{g}$ are homotopic by a fibre preserving homotopy, written $\tilde{f} \simeq \tilde{g}$.*

It will be convenient to consider an altered form of the above lemma.

LEMMA 2. *There is a map $\tilde{s}: E \to E_\infty$, which properly covers a classifying map $s: B \to B_\infty$ for the fibration $p: E \to B$, having the property that $\tilde{s}h \simeq \tilde{s}$ for any fibre homotopy equivalence $h: E \to E$.*

Please note that the fibre preserving homotopy between $\tilde{s}h$ and $\tilde{s}$ is not necessarily a fibre-wise homotopy, i.e., the homotopy need not cover s for all values of $t \in I$.

Proof that Lemma 2 $\Rightarrow$ *Lemma* 1. We shall prove that $\tilde{f} \simeq \tilde{s}$. The argument for $\tilde{g} \simeq \tilde{s}$ is exactly the same, so $\tilde{f} \simeq \tilde{g}$.

Both f and $s: B \to B_\infty$ are classifying maps for $p: E \to B$, so f is homotopic to s. By the covering homotopy property, there is a $\tilde{s}_1: E \to E_\infty$ such that $\tilde{s}_1$

properly covers f and $\tilde{s}_1 \simeq \tilde{s}$.

Let f^*E_∞ be the total space of the fibration induced by f. Then we obtain the following diagram:

$$\begin{array}{ccccc} E & \overset{\bar{f}}{\underset{\bar{s}_1}{\rightrightarrows}} & f^*E_\infty & \xrightarrow{\ \eta\ } & E_\infty \\ \downarrow{\scriptstyle p} & & \downarrow & & \downarrow{\scriptstyle p_\infty} \\ B & \xrightarrow[1_B]{} & B & \xrightarrow[f]{} & B_\infty \end{array}$$

Here η is the natural map $f^*E_\infty \rightarrow E_\infty$ and $\bar{f}$ and $\bar{s}_1$ are the unique maps such that $\eta\bar{f} = \tilde{f}$ and $\eta\bar{s}_1 = \tilde{s}_1$.

Now $\bar{f}$ and $\bar{s}_1$ both properly cover 1_B, and so, by a result of Dold, $\bar{f}$ and $\bar{s}_1$ are fibre homotopy equivalences. Let $\bar{f}^{-1}$, $\bar{s}_1^{-1}$: $f^*E_\infty \rightarrow E$ be fibre homotopy inverses to $\bar{f}$ and $\bar{s}_1$ respectively. Now $\bar{s}_1^{-1} \circ \bar{f}$ is a fibre homotopy equivalence from E to E, so by Lemma 2 and the above considerations,

$$\tilde{s} \simeq \tilde{s} \circ (\bar{s}_1^{-1} \circ \bar{f}) \simeq \tilde{s}_1 \circ (\bar{s}_1^{-1} \circ \bar{f}) = \eta \circ \bar{s}_1 \circ \bar{s}_1^{-1} \circ \bar{f} \simeq \eta \circ \bar{f} = \tilde{f}\,.$$

PROOF OF LEMMA 2. To show the existence of such an $\tilde{s}$, we will construct a fibration using the techniques of Dold [2, pp. 6.3–6.5]. Let J_1 and J_2 be two closed intervals whose union is the circle and whose intersection consists of two disjoint small intervals. Then $p \times 1$: $E \times J_\nu \rightarrow B \times J_\nu$, $\nu = 1, 2$ are fibrations. Here we let $H_\nu = E \times J_\nu$ and $X_\nu = B \times J_\nu$ and $U = B \times (J_1 \cap J_2)$. Then, in the notation of Dold, $H_1^U = H_2^U = E \times (J_1 \cap J_2)$. Any fibre homotopy equivalence h: $E \rightarrow E$ gives rise to a fibre homotopy equivalence φ: $H_1^U \rightarrow H_2^U$ by defining $\varphi = h \times 1$ over one component of U and $\varphi =$ identity over the other component. Using this φ we construct R as in [2, p. 6.4]. Defining $H_1 \cup H_2 \cup R$ as in [2], we obtain a weak fibration, which we shall denote by $E_\varphi \rightarrow B \times S^1$.

We can regard E as a subspace of E_φ under an inclusion map $\tilde{i}$ which takes $e \rightarrow (e, r)$ where $r \notin J_1 \cap J_2$. Then we can define a fibre preserving homotopy h_t: $E \rightarrow E_\varphi$ such that $h_0 = \tilde{i}$ and $h_1 = \tilde{i} \circ h$.

Now from each fibre homotopy equivalence class, we select a fibre homotopy equivalence and perform the above constructions. Then we identify the subsets E of each of these E_{φ_i} together. Call the space M. There results a weak fibration $M \rightarrow B \times (\bigvee S^1)$. Since every weak fibration is fibre homotopy equivalent to a Hurewicz fibration, we obtain the following diagram:

$$\begin{array}{ccccccc} E & \xrightarrow{\ \tilde{i}\ } & M & \xrightarrow{\ \tilde{j}\ } & M^* & \xrightarrow{\ \tilde{k}\ } & E_\infty \\ \downarrow{\scriptstyle p} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle p_\infty} \\ B & \xrightarrow[i]{} & B \times (\bigvee S^1) & \xleftrightarrow[1]{} & B \times (\bigvee S^1) & \xrightarrow[k]{} & B_\infty \end{array}$$

Here M^* is the fibre space and $\tilde{j}$ is a fibre homotopy equivalence and k is the classifying map of M^* and $\tilde{k}$ properly covers k. Then we define our map $\tilde{s} = \tilde{k}\tilde{j}\tilde{i}$. Then any fibre homotopy equivalence $h'\colon E \to E$ gives rise to a fibre preserving homotopy

$$E \xrightarrow{h'_t} E_{\varphi'} \subset M \xrightarrow{\tilde{j}} M^* \xrightarrow{\tilde{k}} E_\infty$$

connecting $\tilde{s}$ with $\tilde{s} \circ h'$.

Lemma 1 $\Rightarrow$ *Lemma* 2 *of* [1]. Let $p\colon E \to B$ be a fibration and let $k\colon B \to B_\infty$ be a classifying map for p. Suppose $\tilde{k}\colon E \to E_\infty$ properly covers k. Let $i\colon X \to CX$ be the inclusion of X into the cone over X. Then we have the commutative diagram

$$\begin{array}{ccccc} E \times X & \xrightarrow{1\times i} & E \times CX & \xrightarrow{\tilde{k}\times \text{const.}} & E_\infty \\ \downarrow{\scriptstyle p\times 1} & & \downarrow{\scriptstyle p\times 1} & & \downarrow{\scriptstyle p_\infty} \\ B \times X & \xrightarrow[1\times i]{} & B \times CX & \xrightarrow[k\times \text{const.}]{} & B_\infty \end{array}$$

To prove Lemma 2 of [1], we must show that for any map $\tilde{A}\colon E \times X \to E_\infty$ which properly covers a map $A\colon B \times X \to B_\infty$, we have $\tilde{A} \simeq (\tilde{k} \times \text{const.}) \circ (1 \times i)$. This follows from Lemma 1 and the fact that both A and $(k \times \text{const.}) \circ (1 \times i)$ are classifying maps of $p \times 1\colon E \times X \to B \times X$.

A remark in [1, § 6] should be amended. If $E_* \to B_*$ is a universal fibre bundle or fibre space for some theory, *and if* $E_* \to B_*$ *satisfies the proper analogue of Lemma* 1, then the analogues of the results of [1] will follow.

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(Received May 1, 1968)
(Revised December 24, 1968)