EVALUATION SUBGROUPS OF HOMOTOPY GROUPS.

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Introduction. In this paper we shall define and study the evaluation subgroups, $G_n(X)$, of a topological space X. This extends and generalizes the author's work on $G_1(X)$ which appears in [9], [10] and [11]. We obtain some interesting geometric corollaries as a result of our investigation. For example. if $p: E \to S^3$ is a fibration with fibre F a finite dimensional CW complex such that $\pi_2(F)$ has no torsion elements, then $p: E \to S^n$ admits a cross-section.

The paper is divided into eight sections. In Section 1 we define the subgroup $G_n(X)$, which we call the nth evaluation subgroup of $\pi_n(X)$. The relationship between evaluation maps from mapping spaces to X and $G_n(X)$ is examined and it is shown that $G_n(X)$ si an invariant of homotopy type in the category of spaces homotopically equivalent to CW complexes.

In Section 2, we study the evaluation subgroup of product spaces and of H-spaces. Then we generalize and note some results proved in [10]. Finally we record the relationship between the evaluation subgroups and the homotopy exact sequence of a fibration which was developed in [11].

Section 3 serves as an introduction to the next two sections. We gather in § 3 Theorem 3-1, proved in [10], and some of its corollaries. The next two sections result in partial generalizations of Theorem 3-1.

In Section 4 we develop the homology structure arising from $G_n(X)$ to obtain in Theorems 4-1 and 4-4 relations between $G_n(X)$, the Euler-Poincaré number of X, and the kernel of the Hurewicz homomorphism.

In Section 5 we investigate the effect of $G_n(X)$ on the cohomology of X and obtain more results using the Hurewicz homomorphism when X has finite dimensional cohomology or X is a suspension of another space. We compute $G_n(S^n)$.

Section 6 is devoted to the relationship of $G_n(\tilde{X})$ to $G_n(X)$, where $\tilde{X}$ is a covering space, or an n-connective covering space, of X.

The results of the earlier sections are applied in Section 7 to compute $G_n(X)$ and apply the results to cross-sections of fibre spaces over spheres.

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One the more interesting results is Theorem 7-1 which states that $G_2(X)$ is contained in the torsion subgroup of $\pi_2(X)$ if X is a finite dimensional CW complex. We find a new proof of the fact that an H-space has Euler Poincaré number equal to zero or is contractible.

In Section 8, various questions and conjectures about the work of this paper are recorded.

1. The evaluation subgroup. Let X be any topological space. Let S^n be the n-sphere. Consider the class of continuous functions

$$F: X \times S^n \to X$$

such that $F(x, s_0) = x$, where $x \in X$ and s_0 is a base point of S^n. Then the map $f: S^n \to X$ defined by $f(s) = F(x_0, s)$, where x_0 is a base point of X, represents an element $\alpha = [f] \in \pi_n(X, x_0)$.

Definition. The set of all elements $\alpha \in \pi_n(X, x_0)$ obtained in the above manner from some F will be denoted by $G_n(X, x_0)$.

Thus for every $\alpha \in G_n(X, x_0)$, there is at least one map $F: X \times S^n \to X$ which satisfies the above conditions such that $[f] = \alpha$. We say that F is an *affiliated map* to α. Note that α may have two affiliated maps which are not homotopic.

It is easy to see that $G_n(X, x_0)$ forms a subgroup of $\pi_n(X, x_0)$. We call $G_n(X, x_0)$ the *n-th evaluation subgroup of* $\pi_n(X, x_0)$. Presently we shall explain why the word evaluation was chosen.

We wish to study the category of spaces which are homotopy equivalent to CW complexes. Thus, from this point on, we shall always assume the hypothesis that X has the homotopy type of a CW complex. The main goal of this section is to show that evaluation subgroups are invariants of homotopy type in this category. This will be done in Theorem 1-7.

Let A and B be CW complexes. Let $L(A, B)$ denote the space of mappings from A to B with the compact open topology. If $* \in A$ is a base point, the map $\omega: L(A, B) \to B$ given by $\omega(f) = f(*)$ is continuous. We call ω an *evaluation* map.

Now ω induces a homomorphism

$$\omega_*: \pi_n(L(A, B), k) \to \pi_n(B, k(*))$$

for all n. Since A is a CW complex, any continuous map $S^n \to L(A, B)$ gives rise to a continuous associated map $A \times S^n \to B$; and conversely, any continuous map $A \times S^n \to B$ is the associated map of a continuous map $S^n \to L(A, B)$. This fact easily establishes the following proposition.

PROPOSITION 1-1. *If X is a CW complex, then*

$$\omega_*[\pi_n(L(X,X),1_X)] = G_n(X,x_0)$$

where $\omega(f) = f(x_0)$.

Because of Proposition 1-1, $G_n(X,x_0)$ is called the "evaluation" subgroup of X.

Of course, one may generalize the problem and note that $\omega_*[\pi_n(L(A,X),k)]$ is also a subgroup of $\pi_n(X,x_0)$ where $k(*) = x_0$. In this setting, $G_n(X,x_0)$ is the lower bound of all the subgroups formed in the above manner. This is made precise in the following proposition.

PROPOSITION 1-2. *If $k(*) = x_0$ and A and X are CW complexes, then*

$$G_n(X,x_0) \subseteq \omega_*[\pi_n(L(A,X),k)]$$

Proof. If $\alpha \in G_n(X,x_0)$, there is an affiliated map $F\colon X \times S^n \to X$. The composition

$$A \times S^n \xrightarrow{k \times 1} X \times S^n \xrightarrow{F} X$$

establishes that $\alpha \in \omega_*[\pi_n(L(A,X),k)]$.

The subgroups $\omega_*[\pi_1(L(A,X),k)]$ of the fundamental group play an important role in fixed point theory. They are called Jaing subgroups in honor of Bo-Ju Jaing who first recognized their importance to fixed point theory (see [14]). Jaing subgroups appear in the work of R. F. Brown and his students, see [3], [5], [6]. Of course, $G_1(X,x_0)$ plays a central role in the study of Jaing subgroups as a result of Proposition 1-2.

In [9], [10], and [11], $G_1(X,x_0)$ is written as $G(X,x_0)$. In [11], $G_1(X,x_0)$ was shown to play a dual role in the theory of Hurewicz fibrations.

From now on, we shall concern ourselves with a series of results which lead to Theorem 1-7.

Our first result will show that, in the usual sense, $G_n(X,x_0)$, viewed as a subgroup of $\pi_n(X,x_0)$, is independent of the base point. Let $\sigma\colon I \to X$ be a path such that $\sigma(0) = x_0$ and $\sigma(1) = x_1 \in X$. Then σ induces a isomorphism $\sigma_*\colon \pi_n(X,x_1) \cong \pi_n(X,x_0)$. See [13], p. 126.

PROPOSITION 1-3. $\sigma_*\colon G_n(X,x_1) \cong G_n(X,x_0)$.

Proof. Define $h_t\colon S^n \to X$ as follows:

Let $\alpha \in G_n(X,x_1)$. Then there exists an affiliated map $F\colon X \times S^n \to X$ such that $F(x,*) = x$ and $F(x_1,y) = f(y)$ where $[f] = \alpha$. Now define

$$h_t(y) = F(\sigma(1-t), y) \text{ for } y \in S^n.$$

It is clear that $h_0 = f$ and h_1 represents $\sigma_*[f] \in \pi_n(X, x_0)$. Now the existence of $F: X \times S^n \to X$ shows that $\sigma_*[f] \in G_n(X, x_0)$.

This proof works for every element of $G_n(X, x_1)$, so we see that $\sigma_*(G_n(X, x_1)) \subseteq G_n(X, x_0)$. On the other hand, we know the reverse path $\sigma^{-1}: I \to X$ induces an inverse isomorphism, $(\sigma^{-1})_*: \pi_n(X, x_0) \to \pi_n(X, x_1)$, to σ_*. Thus $\sigma_*: G_n(X, x_1) \to G_n(X, x_0)$ and $(\sigma^{-1})_*: G_n(X, x_0) \to G_n(X, x_1)$ where σ_* is the inverse of $(\sigma^{-1})_*$. Hence, by definition, $\sigma_*: G_n(X, x_1) \cong G_n(X, x_0)$.

By virtue of this result we frequently write $G_n(X)$ instead of $G_n(X, x_0)$.

It is not true that $f: X \to Y$ induces a map from $G_n(X)$ to $G_n(Y)$. This was shown in [10]. However, for some maps, it is true that f_* maps $G_n(X)$ into $G_n(Y)$. Suppose $r: X \to Y$. We say that r has a *right homotopy inverse* if there is a map $i: Y \to X$ such that $r \circ i$ is homotopic to 1_Y.

PROPOSITION 1-4. *Suppose that Y is a CW complex. If $r: X \to Y$ has a right homotopy inverse $i: Y \to X$, then $r_*: \pi_n(X, x_0) \to \pi_n(Y, r(x_0))$ carries $G_n(X, x_0)$ into $G_n(Y, r(x_0))$.*

Proof. We need two facts which follow from the hypothesis that Y is a CW complex. First, since $Y \times * \subseteq Y \times S^n$ is a subcomplex, $Y \times *$ has the homotopy extension property in $Y \times S^n$. Second, any point in Y has the homotopy extension propetry.

Let $\alpha \in G_n(X, x_0)$ and let $F: X \times S^n \to X$ be an affiliated map to α. We define a map $F': Y \times S^n \to Y$ by letting

$$F'(y, s) = r \circ F(i(y), s).$$

Now $F'(y, *) = r \circ F(i(y), *) = r \circ i(y)$. Since $r \circ i \cong 1_Y$, we may find a homotopy H'_t connecting F' with a map, H'_1, for which $H'_1(y, *) = y$. This follows from the homotopy extension property for $Y \times *$ in $Y \times S^n$.

By the homotopy extension property of points in Y, we may assume that $i(y_0) = x_0$, where $y_0 = r(x_0)$. Then observe that F' restricted to $y_0 \times S^n$ gives a map of $S^n \to Y$ which represents $r_*(\alpha)$. Let $\sigma: I \to Y$ be given by $\sigma(t) = H'_t(y_0, *)$. Then σ induces an isomorphism $\sigma_*: \pi_n(Y, y_0) \to \pi_n(Y, y_0)$. Let $h: S^n \to Y$ be given by restricting H'_1 to $y_0 \times S^n$. Then $\sigma_*[h] = r_*(\alpha)$. Since $[h] \in G_n(Y, y_0)$, then $\sigma_*[h] \in G_n(Y, y_0)$ by Proposition 1-3, so $r_*(\alpha) \in G_n(Y, y_0)$ as was to be proved.

COROLLARY 1-5. *If $r: X \to Y$ is a retract, then*

$$r_*: G_n(X, x_0) \to G_n(Y, r(x_0)).$$

COROLLARY 1-6. *Let Y be a CW-complex. If $i\colon Y \to X$ has a left homotopy inverse, then $i_*(\alpha) \in G_n(X, x_0)$ implies that $\alpha \in G_n(Y, y_0)$ where $i(y_0) = x_0$.*

Proof. By the homotopy extension property for points in Y we may find an $r\colon X \to Y$ such that $r(x_0) = y_0$ and $r \circ i \simeq 1_Y$. Let $h_t\colon Y \to Y$ be the homotopy connecting $r \circ i$ and 1_Y. Let $\sigma\colon I \to Y$ be the closed path given by $\sigma(t) = h_t(y_0)$. Then

$$r_* \circ i_* = \sigma_*\colon \pi_1(Y, y_0) \to \pi_1(Y, y_0).$$

If $i_*(\alpha) \in G_n(X)$, then $r_*(i_*(\alpha)) \in G_n(Y)$ by Proposition 1-4 and hence $\alpha = \sigma_*^{-1}(r_*(i_*(\alpha))) \in G_n(Y)$ by Proposition 1-3.

Now we can prove that $G_n(X)$ is a homotopy type invariant by using 1-4 and 1-6.

THEOREM 1-7. *Suppose that X and Y are both the homotopy type of a CW complex. If $f\colon X \to Y$ is a homotopy equivalence, then f_* carries $G_n(X, x_0)$ isomorphically onto $G_n(Y, f(x_0))$.*

Proof. First we shall assume that Y is a CW complex. Since f has a right homotopy inverse, we have, by Proposition 1-4, that $f_*(G_n(X)) \subseteq G_n(Y)$. Since f has a left homotopy inverse, we have, by Corollary 1-6, $f_*^{-1}(G_n(Y)) \subseteq G_n(X)$. Thus $G_n(Y) = f_* f_*^{-1}(G_n(Y)) \subseteq f_* G_n(X)$. Hence $f_*(G_n(X)) = G_n(Y)$. Since f_* is an isomorphism, the theorem is true for the special case that Y is a CW complex.

Now in general, Y is homotopy equivalent to a CW *complex* Z. Let $g\colon Y \to Z$ be a homotopy equivalence. Then the composition

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is a homotopy equivalence. By the previous paragraph, $g_* f_*$ carries $G_n(X)$ isomorphically onto $G_n(Z)$ and g_* carries $G_n(Y)$ isomorphically onto $G_n(Z)$. Hence f_* must carry $G_n(X)$ isomorphically onto $G_n(Y)$. This proves the theorem.

2. Some elementary formulas. The purpose of this section is to record some properties of $G_n(X)$ which have elementary homotopy theory proofs. We begin by computing $G_n(X)$ when X is a product space or an H-space and we conclude by stating results which were proved elsewhere.

THEOREM 2-1. $G_n(X \times Y, (x_0, y_0)) \cong G_n(X, x_0) \oplus G_n(Y, y_0)$.

Proof. Let $Z = X \times Y$ and $z_0 = (x_0, y_0)$. There exists an isomorphism

$$h\colon \pi_1(Z, z_0) \to \pi_1(X, x_0) \oplus \pi_1(Y, y_0),$$

such that

$$h([\alpha]) = p_*([\alpha]) \oplus q_*([\alpha])$$

where p_* and q_* are induced homomorphisms from the projections of Z onto X and Y respectively. (see [13]). Now $h(G_n(Z)) \subseteq G_n(X, x_0) \oplus G_n(Y, y_0)$ as may readily be seen by noting that p and q are retractions and applying Corollary 1-5.

On the other hand, let $[\alpha]$ and $[\beta]$ be elements of $G_n(X, x_0)$ and $G_n(Y, y_0)$ respectively.

Now $h^{-1}([\alpha] \oplus [\beta]) = [(j \circ \alpha) \cdot (k \circ \beta)]$ where j and k inject $X \to X \times x_0$ and $Y \to Y \times y_0$ respectively.

Let $H\colon X \times S^n \to X$ such that H is affiliated with $[\alpha]$. Suppose we choose α such that $\alpha(s) = H(x_0, s)$. Then define $K\colon X \times Y \times S^n \to X \times Y$ such that

$$K(x, y, s) = (H(x, s), y).$$

The existence of K shows that $[j \circ \alpha] \in G_n(Z, z_0)$. Similarly $[k \circ \beta] \in G_n(Z, z_0)$. Thus the product $[j \circ \alpha] \cdot [k \circ \beta] \in G_n(Z, z_0)$. Therefore

$$h^{-1}(G_n(X) \oplus G_n(Y)) \subseteq G_n(Z).$$

Hence $h(G_n(Z)) \supseteq G_n(X) \oplus G_n(Y)$ and so

$$h(G_n(Z)) = G_n(X) \oplus G_n(Y)$$

as was to be shown.

Now suppose X is an H-space. Then we have the following fact:

PROPOSITION 2-2. *Suppose X is an H-space, then $G_n(X) = \pi_n(X)$.*

Proof. Let e be the identity and let $x \cdot y$ denote x multiplied by y. Then given any map $f\colon S^n \to X$ such that $f(*) = e$, we have a map $F\colon X \times S^n \to X$ such that

$$F(x, s) = x \cdot f(s).$$

Since $F(x, *) = x \cdot e = x$ and $F(e, s) = f(s)$, the existence of F implies that $[f] \in G_n(X)$ for all $f\colon S^n \to X$. This proves the proposition.

In general, $G_n(X) \neq \pi_n(X)$. There does exist a subgroup $P_n(X, x_0)$ of $\pi_n(X, x\ )$ which must contain $G_n(X, x_0)$. We define $P_n(X, x_0)$ to be the set of elements of $\pi_n(X,x_0)$ whose Whitehead product with all elements of all homotopy groups is zero. It turns out that $P_n(X,x_0)$ form a subgroup of $\pi_n(X,x_0)$. If $[f]\in\pi_n(X,x_0)$, a necessary and sufficient condition that $[f]\in P_n(X,x_0)$ is that for every $[g]\in\pi_m(X,x_0)$ and every m, there exists a map $G\colon S^m\times S^n\to X$ such that $G(*,s)=f(s)$ for all $s\in S^n$ and $G(r,*)=g(r)$ for all $r\in S^m$.

PROPOSITION 2.3. $G_n(X,x_0)\subseteq P_n(X,x_0)$.

Proof. Let $\alpha\in G_n(X,x_0)$. Then there is an affiliated map $F\colon X\times S^n\to X$ such that $F(x,*)=x$ and $F(x_0,s)=f(s)$.

Let $[g]\in\pi_m(X,x_0)$. Then the map $G\colon S^m\times S^n\to X$ given by $G(r,s)=F(g(r),s)$, restricted to S^m is g and restricted to S^n is f. Thus $\alpha\in P_n(X,x_0)$.

As a corollary to the above result, we prove a result from [10].

COROLLARY 2-4. $G_1(X,x_0)\subseteq Z(\pi_1(X,x_0))$, *the center of* $\pi_1(X)$.

Proof. If $\alpha\in\pi_1(X)$, then $[\alpha,\beta]=\alpha_*(\beta)-\beta$, where $[\alpha,\beta]$ is the Whitehead product of α with any $\beta\in\pi_m(X)$, and $\alpha_*(\beta)$ is the action of α on β. If $\alpha\in P_1(X)$, then α must operate trivially on $\pi_n(X)$ for all n. The subgroup of $\pi_1(X,x_0)$ which operates trivially on $\pi_1(X)$ is $Z(\pi_1(X,x_0))$, so $P_1(X)\subseteq Z(\pi_1(X))$.

Remark 1. T. Ganea has produced an example in [8] which shows that $G_1(X)\neq P_1(X)$.

Remark 2. Corollary 2-4 is trivially true for $n>1$ since $\pi_n(X)$ is abelian. However, there is no group theoretic proper subgroup of $\pi_n(X)$ which must contain $G_n(X)$ if $n\geqq 2$. That is, for any abelian group π, there is a space X such that $G_n(X)=\pi_n(X)=\pi$. This is seen to be true by allowing X to be a $K(\pi,n)$. Then X is an H-space and the statement follows by Proposition 2-2. When $n=1$, it is possible to find a space X such that $G_1(X)$ is center of any group π. This is seen from the following theorem which is proved in [10].

THEOREM 2.-5. *If X has the homotopy type of an aspherical CW complex, then* $G_1(X)=Z(\pi_1(X))$, *the center of* $\pi_1(X)$.

Let F be the homotopy type of a CW complex. Then there is a universal fibration $p_\infty\colon E_\infty\to B_\infty$ with fibre F_∞ which is homotopy equivalent to F. See [2], [7] or [10] for more details. There is a homomorphism $d_\infty\colon\pi_{n+1}(B_\infty)\to\pi_n(F_\infty)$ which arises from the homotopy exact sequence of the fibration p_∞. The following theorem is proved in [10].

THEOREM 2-6. $G_n(F_\infty) = d_\infty(\pi_{n+1}(B_\infty))$. *Thus, for any fibration*

$$F \xrightarrow{i} E \xrightarrow{p} B,$$

$d(\pi_{n+1}(B)) \subseteq G_n(F)$ *where* $d\colon \pi_{n+1}(B) \to \pi_n(F)$ *arises from the homotopy exact sequence of the fibration.*

Comparing the above theorem with Proposition 1-2, we see that $G_n(X)$ occupies a very strategic position. It is the intersection of all subgroups of $\pi_n(X)$ which are the image of a homomorphism induced by an evaluation map and also the union of all subgroups of $\pi_n(X)$ which are the image of a boundary homomorphism which arise in the fibre homotopy exact sequence of some fibration with X as fibre.

COROLLARY 2-7. *If* $G_n(F) = 0$, *then every fibre space over* S^{n+1}, *with fibre F, has a cross-section.*

This fact follows from the previous remarks and the covering homotopy property. It turns out that it is possible to prove that $G_n(F) = 0$ for a large class of spaces.

3. The Euler-Poincaré number and $G_1(X)$. The following theorem was proved in [10] (Theorem IV. 1), using Nielsen-Wecken fixed point theory.

THEOREM 3-1. *Let X have the homotopy type of a compact polyhedron. If the Euler Poincaré number* $\chi(X) \neq 0$, *then* $G_1(X)$ *is the trivial subgroup.*

This theorem leads to corollaries of algebraic and geometric interest. For example, combining the above theorem with Theorem 2-5 we obtain the following result, (see [10]).

COROLLARY 3-2. *If X is homotopy equivalent to an aspherical, compact polyhedron and* $\chi(X) \neq 0$, *then the center of* $\pi_1(X)$ *is trivial.*

John Stallings has put this result in a homological algebraic setting in [16].

Let F be a connected graph contained in S^3. Papakriakopoulos has shown that $S^3 - F$ is aspherical. By use of this result and Alexander Duality (see Corollary IV. 5, [10]), we can prove the following corollary to Theorem 3-1.

COROLLARY 3-3. *Let F be a connected graph imbedded as a subcomplex* *of S^3. If the center of $\pi_1(S^3-F)$ is nontrivial, then F has the homotopy type of a circle.*

It would be interesting to generalize Theorem 3-1 from $G_1(X)$ to $G_n(X)$ in the hope that we may obtain more, interesting corollaries as we did above. Certainly the statement of the theorem offers many possible ways of generalization, for the statement relies only upon concepts of homotopy theory and homology theory. On the other hand, the proof depends upon Nielsen-Wecken fixed point theory and offers no possibilities of generalization. We need new methods.

In the next two sections, we examine the homology and cohomology which arises in connection with $G_n(X)$ and we are able to find analogous theorems to Theorem 3-1. The most analogous is Theorem 4-1. The hypotheses generalize from compactness to finitely generated homology, but the conclusion specializes for the case $G_1(X)$ to a weaker result.

4. Homology and $G_n(X)$. We wish to study the algebraic structure induced by $\phi\colon X\times S^n\to X$, affiliated to some $\alpha\in G_n(X)$, on the homology and cohomology of X.

By the Kunneth formula and the fact that $H_*(S^n;Z)$ has no torsion, we have

$$H_*(X\times S^n;G)\cong H_*(X;G)\otimes H_*(S^n;Z).$$

Thus if $x\in H_*(X\times S^n;G)$, we may represent $x=y\otimes 1+z\otimes\lambda$ where $\lambda\in H_n(S^n;Z)$ is a fundamental class of S^n. Define $i_1\colon X\to X\times S^n$ by $i_1(x)=(x,*)$. Define $i_2\colon S^n\to X\times S^n$ by $i_2(s)=(*,s)$.

Let p_1 be projection from $X\times S^n$ to X and let p_2 be projection from $X\times S^n$ to S^n. Then, p. 235 [15], $p_{1*}(z\otimes z')=z\otimes\epsilon(z')$, where ϵ is the augmentation. That is,

$$p_{1*}(x\otimes 1)=x \text{ and } p_{1*}(x\otimes\lambda)=0.$$

Also $p_{2*}(1\otimes\lambda)=\lambda$, and $p_{2*}(x\otimes\lambda)=0$ and $p_{2*}(x\otimes 1)=0$ if $x\in H_q(X;G)$ where $q>0$.

Now since $p_1i_1=1_X$, $i_{1*}(x)=x\otimes 1$ for $x\in H_*(X;G)$ and since $p_2i_2=1_{S^n}$, $i_{2*}(y)=1\otimes y$, $y\in H_*(S^n;Z)$. So since $1_X=\phi\circ i_1$, $\phi_*(x\otimes 1)=x$ and since $f=\phi\circ i_2$, $\phi_*(1\otimes\lambda)=f_*(\lambda)$, where $f\colon S^n\to X$ is induced by $\phi\colon X\times S^n\to X$.

Let $x\in H_q(X;G)$. We shall define $x\lambda\in H_{q+n}(X;G)$ to be equal to $\phi_*(x\otimes\lambda)$. It is easily seen that $\lambda\colon H_q(X;G)\to H_{q+n}(X;G)$ such that $x\to x\lambda$ is a homomorphism. We shall define $x\lambda^n=(x\lambda^{n-1})\lambda$.

Let ψ stand for the diagonal map for any space $Y \to Y \times Y$.

$$\begin{tikzcd}
 & (X \times X) \times (S^n \times S^n) \arrow[d, "1 \times T \times 1"] \\
X \times S^n \arrow[ur, "\psi \times \psi"] \arrow[r, "\psi"] \arrow[d, "\phi"] & (X \times S^n) \times (X \times S^n) \arrow[d, "\phi \times \phi"] \\
X \arrow[r, "\psi"] & X \times X
\end{tikzcd}$$

The above diagram commutes where $T\colon X \times S^n \to S^n \times X$ is given by $T(x,s) = (s,x)$.

Let G be a field. Then by the Kunneth formula,

$$H_*(X;G) \otimes H_*(X;G) \cong H_*(X \times X;G).$$

Now $T_*\colon H_*(X;G) \otimes H_*(S^n;Z) \to H_*(S^n;Z) \otimes H_*(X;G)$ by letting $T_*(z \otimes x) = (-1)^{pq} x \otimes z$ where $z \in H_p(X;G)$ and $x \in H_q(S^n;Z)$. Thus

$$\begin{aligned}
\psi_*(x\lambda) &= \psi_* \phi_* (x \otimes \lambda) \\
&= (\phi_* \otimes \phi_*) \circ (1 \otimes T_* \otimes 1) \circ (\psi_* \otimes \psi_*)(x \otimes \lambda) \\
&= (\phi_* \otimes \phi_*) \circ (1 \otimes T_* \otimes 1)(\psi_*(x) \otimes (1 \otimes \lambda + \lambda \otimes 1)).
\end{aligned}$$

Now $\psi_*(x)$ has the form $\Sigma z_i \otimes z_i'$ where $z_i \in H_*(X;G)$ and $z_i' \in H_*(X;G)$ so

$$\begin{aligned}
\psi_*(x\lambda) &= (\phi_* \otimes \phi_*)\{\Sigma(z_i \otimes 1) \otimes (z_i' \otimes \lambda) \\
&\qquad + \Sigma(-1)^{n \dim z_i'}(z_i \otimes \lambda) \otimes (z_i' \otimes 1)\} \\
&= \Sigma z_i \otimes z_i'\lambda + \Sigma(-1)^{n \dim z_i'} z_i\lambda \otimes z_i'.
\end{aligned}$$

We shall establish the following convention. We regard the symbols $\lambda^p \otimes \lambda^q$, $p, q = 0, 1, 2, \cdots$, as right operators on $H_*(X;G) \otimes H_*(X;G)$ by the rule

$$(x \otimes y)(\lambda^p \otimes \lambda^q) = (-1)^{pn \cdot \dim y} x\lambda^p \otimes y\lambda^q.$$

Thus, by the above remarks, we see that

$$\psi_*(x\lambda) = \psi_*(x)(\lambda \otimes 1 + 1 \otimes \lambda)$$

Note that

$$\begin{aligned}
\psi_*(x\lambda^2) &= \psi_*(x\lambda)(\lambda \otimes 1 + 1 \otimes \lambda) \\
&= \{\psi_*(x)(\lambda \otimes 1 + 1 \otimes \lambda)\}(\lambda \otimes 1 + 1 \otimes \lambda).
\end{aligned}$$

So $\psi_*(x\lambda^p) = (\psi_* x)(\lambda\otimes 1 + 1\otimes\lambda)^p$ where

$$(\lambda\otimes 1 + 1\otimes\lambda)^p = (\lambda\otimes 1 + 1\otimes\lambda)^{p-1}(\lambda\otimes 1 + 1\otimes\lambda).$$

We have two cases to consider. If n is even, then $(x\otimes y)(\lambda^p\otimes\lambda^q) = x\lambda^p\otimes y\lambda^q$, so we may regard

$$(\lambda\otimes 1 + 1\otimes\lambda)^p = \sum_{i=0}^{p} \binom{p}{i} \lambda^i\otimes\lambda^{p-i}.$$

On the other hand, if n is odd, then observe that

$$\begin{aligned}(x\otimes y)(\lambda\otimes 1 + 1\otimes\lambda)^2 &= (x\otimes y)(\lambda\otimes 1 + 1\otimes\lambda)(\lambda\otimes 1 + 1\otimes\lambda)\\ &= ((-1)^{n\cdot\dim y} x\lambda\otimes y + x\otimes y\lambda)(1\otimes\lambda + \lambda\otimes 1)\\ &= (-1)^{n\dim y} x\lambda\otimes y\lambda + x\otimes y\lambda^2 + (-1)^{2n\dim y} x\lambda^2\otimes y\\ &\quad + (-1)^{n\dim y\lambda} x\lambda\otimes y\lambda.\end{aligned}$$

Now $\dim y\lambda = \dim y + n$. So $n\dim y\lambda = n\dim y + n^2$. Since n^2 is odd, we have $(-1)^{n\dim y\lambda} = -(-1)^{n\dim y}$, so the $x\lambda\otimes y\lambda$ terms cancel. Thus

$$\begin{aligned}(x\otimes y)(\lambda\otimes 1 + 1\otimes\lambda)^2 &= (x\lambda^2\otimes y + x\otimes y\lambda^2)\\ &= (x\otimes y)(\lambda^2\otimes 1 + 1\otimes\lambda^2).\end{aligned}$$

We shall use these formulas to establish some theorems about $G_n(X)$.

Let $h\colon \pi_n(X)\to H_n(X;Z)$ be the Hurewicz homomorphism. We shall define $h_p\colon \pi_n(X)\to H_n(X;Z)\to H_n(X;Z_p)$ as composition of h tensored with Z_p. h_p will be called the mod p Hurewicz homomorphism. We shall let h_∞ stand for the Hurewicz map $h_\infty\colon \pi_n(X)\to H_n(X;R)$ where R is the field of rationals.

THEOREM 4-1. *Let X be a topological space with finitely generated integer homology. If n is an odd integer, then $G_n(X)\subseteq$ kernel of h_p for p any prime number or ∞ provided the Euler-Poincaré number $\chi(X)\neq 0$.*

Proof. Suppose that $\alpha\in G_n(X)$ is not contained in the kernel of h_p. Then, if $\phi\colon X\times S^n\to X$ is affiliated to α, we have, using the notation above, that $(1)\lambda\neq 0\in H_n(X;Z_p)$. We shall write λ for $(1)\lambda$ when no danger of confusion arises. Suppose $x\in H_i(X;Z_p)$. We say that x has *depth* d if there exists an element $y\in H_{i-dn}(X;Z_p)$ such that $y\lambda^d = x$ and $z\lambda^{d+1}\neq x$ for any $z\in H_*(X;Z_p)$. We prove the following lemma.

LEMMA 4-2. *If $x\lambda = 0$, then x has odd depth.*

Proof. Suppose x has depth d and $y\lambda^d = x$. Then $0 = \psi_*(x\lambda) = (\psi_*(y\lambda^{d+1}))$. Let us assume that d is even and that

$$\psi_*(y) = y\otimes 1 + 1\otimes y + \sum_i y_i\otimes y_i'.$$

Then by remarks above,

$$\begin{aligned}\psi_*(y\lambda^{d+1}) &= (y\otimes 1)(\lambda^2\otimes 1+1\otimes\lambda^2)^{d/2}(1\otimes\lambda+\lambda\otimes 1)\\ &\quad+(1\otimes y)(\lambda^2\otimes 1+1\otimes\lambda^2)^{d/2}(1\otimes\lambda+\lambda\otimes 1)\\ &\quad+\Sigma\pm(y_i\otimes y_i')(\lambda^2\otimes 1+1\otimes\lambda^2)^{d/2}(1\otimes\lambda+\lambda\otimes 1)\\ &=0.\end{aligned}$$

Thus we see that $y\lambda^d\otimes\lambda+\Sigma z_i\otimes z_i'=0$ where dimension of z_i' is n and dimension of z_i is $=\dim y\lambda^d$. Thus $\Sigma z_i\otimes z_i'=\Sigma v_i\lambda^{d+1}\otimes z_i'$ for some $v_i\in H_*(X;Z_p)$. Now $\Sigma v_i\lambda^{d+1}\otimes z_i'=-y\lambda^d\otimes\lambda$. Thus $\Sigma v_i\lambda^{d+1}\otimes z_i'$ must have the form $z\lambda^{d+1}\otimes\lambda$ for some z. Hence $-z\lambda^{d+1}=y\lambda^d=x$, so x has depth $d+1$, a contradiction.

Now $H_q(X;Z_p)$ is a vector space over Z_p and can be written as the direct sum of spaces

$$A_q{}^d\oplus\cdots\oplus A_q{}^0$$

such that $x\in A_q{}^r$ has depth r. We shall let $\lambda(A_q{}^r)$ represent the image of $A_q{}^r$ under λ in $H_{q+n}(X;Z_p)$. Then we may regard

$$H_{q+n}(X;Z_p)=A_{q+n}{}^{d+1}\oplus\cdots\oplus A_{q+n}{}^0$$

and we may require in addition that

$$\lambda(A_q{}^{d-i})\supseteq A_{q+n}{}^{d-i+1}.$$

In fact, let K be the subspace of $A_q{}^{d-i}$ such that every $x\in K$ is mapped onto an element $x\lambda$ with depth greater than $d-i+1$. Then $A_q{}^{d-i}=K\oplus Q$, where Q is a complementary subspace to K. Then define $\lambda(Q)=A_{q+n}{}^{d-i+1}$.

We may inductively define the $A_q{}^d$ such that $\lambda(A_q{}^d)\supseteq A_{q+n}{}^{d+1}$ for all q and d. Now if d is even, the lemma above tells us that $\lambda\colon A_q{}^d\cong A_{q+n}{}^{d+1}$. For suppose not. Then there exists an $x\in A_q{}^d$ such that $x\lambda$ has depth greater than $d+1$. Thus there is a y with depth greater than d such that $y\lambda=x\lambda$. Now $x-y$ has depth d since x has depth d. But $(x-y)\lambda=0$. Since d is even, $x-y=0$. So y has depth d which is a contradiction. Thus λ is the required isomorphism.

Let $\chi(H_*(X;Z_p))=\sum_i(-1)^i(\dim H_i(X;Z_p))$. Since $H_*(X;Z_p)$ is finite dimensional, $\chi(H_*(X;Z_p))$ is well defined. Now the above paragraph implies that $\chi(H_*(X;Z_p))=0$ since

$$\begin{aligned}\chi(H_*(X;Z_p)) &= \sum_q(-1)^q(\sum_d\dim A_q{}^d)\\ &= \sum_q(-1)^q(\sum_{2d}(\dim A_q{}^{2d}+(-1)^n\dim A_{q+n}{}^{2d+1})).\end{aligned}$$

Since $\dim A_q{}^{2d}=\dim A_{q+n}{}^{2d+1}$ and n is odd, $\chi(H_*(X;Z_p))=0$.

Our theorem will be proved by applying the following lemma:

LEMMA 4-3. *Let $H_*(X;Z)$ be finitely generated. Then the Euler-Poincaré number of X, $\chi(X)$, is equal to $\chi(H_*(X;Z_p))$.*

Proof. From the universal coefficient theorem, we see that

$$H_q(X;Z_p) \cong (H(X) \otimes Z_p) \oplus (H_{q-1}(X)*Z_p).$$

See p. 222, [15]. Now $H_{q-1}(X)*Z_p \cong$ kernel of $p\colon H_{q-1}(X) \to H_{q-1}(X)$ where $p\colon x \to px$. Now every cyclic subgroup whose order is divisible by p gives a one dimensional contribution to $H_q(X) \otimes Z_p$ in $H_q((X);Z_p)$ and contributes one dimension to $H_q(X)*Z_p$ in $H_{q+1}(X;Z_p)$. These two contributions cancel out in $\chi(H_q(X;Z_p))$. On the other hand, any infinite cyclic group Z gives a one dimensional contribution to $H_q(X) \otimes Z_p$ and no contribution to $H_{q+1}(X)*Z_p$. So $\chi(H_*(X;Z_p)) = \chi(H_*(X) \otimes R) = \chi(X)$.

THEOREM 4-4. *Let X have finitely generated integer homology. Suppose p is a prime which does not divide $\chi(X)$. Then $G_n(X) \subseteq$ kernel h_p for even n.*

Proof. We shall assume that $\alpha \in G_n(X)$ is not in the kernel of h_p. Now suppose that $\phi\colon X \times S^n \to X$ is affiliated with α, then $(1)\lambda \neq 0 \in H_n(X;Z_p)$.

Now $\psi_*(x\lambda^q) = (\psi_*(x))(\sum_i \binom{q}{i} \lambda^i \otimes \lambda^{q-i})$ for any $x \in H_*(X;Z_p)$ since n is even. We shall show that if $x\lambda = 0$, then x has depth $d \equiv -1 \pmod p$. Then by an argument similar to the last theorem, we can show that $\chi(X)$ is divisible by p.

LEMMA 4-5. *If $x\lambda = 0$, then the depth of x $d \equiv -1 \pmod p$.*

Proof. Suppose $y\lambda^d = x$. Then

$$0 = \psi_*(x\lambda) = \psi_*(y\lambda^{d+1}) = \psi_*(y)(\sum_i \binom{d+1}{i} \lambda^i \otimes \lambda^{d+1-i})$$

Now $\psi_*(y) = y \otimes 1 +$ other terms, so $(d+1)y\lambda^d \otimes 1\lambda$ must appear in $\psi_*(y\lambda^{d+1})$. Now since $\psi_*(y\lambda^{d+1}) = 0$,

$$(d+1)y\lambda^d \otimes 1\lambda + \sum z_i \otimes z_i' = 0$$

where $\sum z_i \otimes z_i'$ consists of all terms in $\psi_*(y\lambda^{d+1})$ such that z_i' has dimension n. Thus $\sum z_i \otimes z_i'$ must come from terms of the form

$$(\sum y_i \otimes z_i')((d+1)\lambda^{d+1} \otimes 1)$$

where $\sum y_i \otimes z_i'$ is the sum of all terms in $\psi_*(y)$ with the z_i' having dimension n. Thus $(d+1)y\lambda^d \otimes 1\lambda + (d+1)\sum y_i\lambda^{d+1} \otimes z_i' = 0$. This implies that a linear combination of the $y_i\lambda^{d+1}$, times $d+1$, equals $(d+1)y\lambda^d$. Hence there exists a z such that $(d+1)y\lambda^d = (d+1)z\lambda^{d+1}$. If $d+1 \neq 0 \pmod p$, then $x = y\lambda^d = z\lambda^{d+1}$, which is a contradiction to the fact that y has depth d and so $d \equiv -1 \pmod p$.

Now, as in the proof of the preceding theorem

$$H_q(X;Z_p) \cong A_q{}^d \oplus \cdots \oplus A_p{}^0$$

such that $\lambda(A_q{}^d) \supseteq A_{q+n}{}^{d+1}$. Now, by the lemma, $\lambda\colon A_q{}^d \cong A_{q+n}{}^{d+1}$ if $d+1$ is not a multiple of p.

Now

$$\begin{aligned}
\chi(H_*(X;Z_p)) &= \sum_q (-1)^q \dim H_q(X;Z_p) \\
&= \sum_q (-1)^q (\sum_d \dim A_q{}^d) \\
&= \sum_q \{(-1)^q \dim A_q{}^0 + (-1)^{q+n} \dim A^1{}_{q+n} + \cdots + \dim A^d{}_{q+dn} + \cdots\} \\
&= \sum_q (-1)^q (\sum_{d=0} \dim A^d{}_{q+dn}) \text{ since } n \text{ is even.}
\end{aligned}$$

Now since $\dim A_q{}^{kp} = \dim A_{q+n}{}^{kp+1} = \cdots = \dim A_{q+(p-1)n}{}^{kp-p+1}$ we see that $\sum_d \dim A^d{}_{q+dn}$ is a multiple of p and so $\chi(H_*(X;Z_p))$ is a multiple of p. Hence, by Lemma 4-3, $\chi(X)$ is a multiple of p.

COROLLARY 4-6. *If X has finitely generated integer homology and if $\chi(X) = 1$, then $G_n(X) \subseteq \ker h_p$ for all n and prime p.*

Proof. No prime divides 1.

5. Cohomology and $G_n(X)$. We now study the consequences in cohomology of the map $\phi\colon X \times S^n \to X$. By the Kunneth formula and the fact that $H^*(S^n;Z)$ has no torsion, we have

$$H^*(X \times S^n; G) \cong H^*(X;G) \otimes H^*(S^n;Z).$$

Thus if $x \in H^*(X \times S^n; G)$, we may write $x = y \otimes 1 + z \otimes \bar{\lambda}$ where $\bar{\lambda} \in H^n(S^n;Z)$ is a fundamental class of S^n dual to λ. Let $i_1\colon X \to X \times S^n$ by $i_1(x) = (x,*)$. Let $i_2\colon S^n \to X \times S^n$ by $i_2(s) = (*,s)$. Let p_1 be projection from $X \times S^n$ to X and let p_2 be projection from $X \times S^n$ to S^n. Then, p. 249 [15], $p_1{}^*(x) = x \otimes 1$. Also $p_2{}^*(\bar{\lambda}) = 1 \otimes \bar{\lambda}$ and of course $p_2{}^*(1) = 1 \otimes 1$.

Now since $p_i i_1 = 1_X$, $i_1{}^*(z \otimes z') = 0$ unless $z' \in H^0(S^n;Z)$ in which case $i_1{}^*(z \otimes 1) = z$. Similarly $i_2{}^*(z \otimes z') = 0$ unless $z \in H^0(X;G)$ and $i_2{}^*(1 \otimes z')$

$= z'$. So since $1_X = \phi \circ i_1$, $\phi^*(x) = x \otimes 1 + y \otimes \bar{\lambda}$. From now on, we shall denote y by $x\bar{\lambda}$, so that $\phi^*(x) = x \otimes 1 + x\bar{\lambda} \otimes \bar{\lambda}$ for all $x \in H^*(X; G)$. The map $x \to x\bar{\lambda}$ is a homomorphism as is easily seen. We define $x\bar{\lambda}^n = (x\bar{\lambda}^{n-1})\bar{\lambda}$.

Now $\bar{\lambda}\colon H^q(X; G) \to H^{q-n}(X; G)$ is dual to $\lambda\colon H_q(X; G) \to H_{q+n}(X; G)$. That is, $\langle u, v\lambda\rangle = \langle u\bar{\lambda}, v\rangle$ where $\langle u, v\rangle$ is the Kronecker product. In fact

$$\begin{aligned}\langle u, v\lambda\rangle &= \langle u, \phi_*(v \otimes \lambda)\rangle = \langle \phi^*(u), v \otimes \lambda\rangle \\ &= \langle u \otimes 1 + u\bar{\lambda} \otimes \bar{\lambda}, v \otimes \lambda\rangle \\ &= \langle u\bar{\lambda} \otimes \bar{\lambda}, v \otimes \lambda\rangle = (-1)^{q(n-n)}\langle u\bar{\lambda}, v\rangle \cdot \langle \bar{\lambda}, \lambda\rangle \\ &= \langle u\bar{\lambda}, v\rangle.\end{aligned}$$

The cup product in $X \times S^n$ is given by

$$(x \otimes y) \cup (x' \otimes y') = (-1)^{qr}(x \cup x' \otimes y \cup y')$$

where $x' \in H^q(X; G)$ and $y \in H^r(S^n; Z)$. Thus

$$\begin{aligned}\phi^*(u \cup v) &= (u \otimes 1 + u\bar{\lambda} \otimes \bar{\lambda}) \cup (v \otimes 1 + v\bar{\lambda} \otimes \bar{\lambda}) \\ &= u \cup v \otimes 1 + (u \cup (v\bar{\lambda}) + (-1)^{n \dim v} u\bar{\lambda} \cup v) \otimes \bar{\lambda}.\end{aligned}$$

The map $f\colon S^n \to X$ given by $f = \phi \circ i_2$ plays a role. Let $x \in H^n(X; G)$. Then

$$\begin{aligned}f^*(x) &= i_2 \circ \phi^*(x) = i_2{}^*(x \otimes 1 + x\bar{\lambda} \otimes \bar{\lambda}) \\ &= i_2{}^*(x\bar{\lambda} \otimes \bar{\lambda}) = r\bar{\lambda}\end{aligned}$$

r is some integer. By dimensional arguments, $x\bar{\lambda} = s \cdot 1$ where s is some integer. Now $i_2{}^*(s \cdot 1 \otimes \bar{\lambda}) = i_2{}^*(1 \otimes s\bar{\lambda}) = s\bar{\lambda} = r\bar{\lambda}$, so $x\bar{\lambda} = r1$.

Now we shall investigate some consequences which flow from the above considerations.

THEOREM 5-1. *Suppose X has only a finite number of nonzero homology groups, then $G_{2n}(X) \subseteq \ker h_\infty$.*

Proof. Suppose $\alpha \in G_{2n}(X)$ is not contained in the kernel of h_∞. Suppose $\phi\colon X \times S^n \to X$ is affiliated with α. Then $h_\infty(\alpha) = (1)\lambda \in H_n(X; R)$ R is the field of rational numbers. Let $\bar{\beta} \in H^{2n}(X; R)$ be dual to $(1)\lambda$. Then $1 = \langle \bar{\beta}, (1)\lambda\rangle = \langle \bar{\beta}\bar{\lambda}, 1\rangle$, so $\bar{\beta}\bar{\lambda} = 1 \in H^0(X; R)$.

Now we shall prove that $\bar{\beta}^r \neq 0$ for all integers r, where $\bar{\beta}^r$ is the cup product of $\bar{\beta}$ with itself r times. First we show that

$$\phi^*(\bar{\beta}^r) = \bar{\beta}^r \otimes 1 + r\bar{\beta}^{r-1} \otimes \bar{\lambda}.$$

This formula is clearly true for $r = 1$ since $\bar{\beta}\bar{\lambda} = 1$. Now suppose it is true for $r - 1$. Then

$$\begin{aligned}\phi^*(\bar\beta^r) &= \phi^*(\beta^{r-1})\phi^*(\bar\beta)\\ &= (\bar\beta^{r-1}\otimes 1 + (r-1)\bar\beta^{r-2}\otimes\bar\lambda)\cup(\bar\beta\otimes 1 + 1\otimes\bar\lambda)\\ &= \bar\beta^r\otimes 1 + ((-1)^{(2n)^2}(r-1)\bar\beta^{r-1} + \bar\beta^{r-1})\otimes\bar\lambda\\ &= \bar\beta^r\otimes 1 + r\bar\beta^{r-1}\otimes\bar\lambda.\end{aligned}$$

Now if $\bar\beta^r = 0$, then $\phi^*(\bar\beta^r) = \bar\beta\otimes 1 + r\bar\beta^{r-1}\otimes\bar\lambda = 0$. So $r\bar\beta^{r-1} = 0$. Thus $\bar\beta^{r-1} = 0$ since we have rational coefficients, but this leads to a contradiction. Since $\bar\beta^r \neq 0$ for all r, $H^{2rn}(X;R)\neq 0$ for all r and hence by the relation between homology and cohomology, X has an infinite number of nonzero homology groups.

The proof of the theorem breaks down if we try the same method for Z_p coefficients since $\bar\beta^{p-1} = 0$ does not imply $\bar\beta^{p-1} = 0$.

However if $G_n(X)\not\subset \ker h_p$, the cohomology structure of X must have many properties. We can exploit this fact in the following theorem:

THEOREM 5-2. *Let X be the suspension of some CW complex Y and suppose that X has integer homology of finite type. If there is an $\alpha\in G_n(X)$ such that $h(\alpha)\neq 0\in H_n(X;Z)$, then $h(\alpha)$ has infinite order, n is odd and X has the rational cohomology of S^n.*

Proof. Since $H_n(X;Z)$ is finitely generated,

$$H_n(X;Z) = F\oplus A_1\oplus\cdots\oplus A_m$$

where F is free abelian group and the A_i are cyclic subgroups generated by $a_i\in A_i$ with order r_i such that $r_i \mid r_{i+1}$ for $1\leqq i\leqq m$. Now assume that $h(\alpha)$ has finite order; then $h(\alpha) = t_1a_1 + \cdots + t_ma_m$ where the t_i are integers such that $0\leqq t_i < r_i$. Since $H_*(X;Z)$ has finite type, the torsion subgroup of $H_n(X;Z)$ is isomorphic to the torsion subgroup of $H^{n+1}(X;Z)$. Thus there is a $\bar b_m\in H^{n+1}(X;Z)$ which corresponds to a_m, and since the torsion subgroup of $H^1(X;Z)$ is trivial, we see that $\bar b_m\bar\lambda = 0\in H^1(X;Z)$. Now suppose k is the largest integer such that $t_k\neq 0$. We shall consider $H_n(X;Z_{r_k})$. Then by the universal coefficient theorem, we have the following split exact sequence

$$0\to H_n(X;Z)\otimes Z_{r_k}\xrightarrow{\mu} H_n(X;Z_{r_k})\to H_{n-1}(X;Z)*Z_{r_k}\to 0$$

We shall let the symbols a_i represent $\mu(a_i\otimes 1)$ in $H_n(X;Z_{r_k})$. Similarly for $h(\alpha)\in H_n(X;Z_{r_k})$.

For cohomology, we have the functorial short exact sequence

$$0\to H^q(X;Z)\otimes Z_{r_k}\xrightarrow{\mu} H^q(X;Z_{r_k})\to H^{q+1}(X;Z_{r_k})*Z_{r_k}\to 0$$

and we define $\bar{b}_m \in H^{n+1}(X; Z_{r_k})$ as $\mu(\bar{b}_m \otimes 1)$. Since the exact sequence is functorial, we have $\bar{b}_m \lambda = 0 \in H^1(X; Z_{r_k})$.

Now we have the following split exact sequence relating holomogy and cohomology:

$$0 \to \operatorname{Ext}(H_{q-1}(X; Z_{r_k}), Z_{r_k}) \to H^q(X; Z_{r_k}) \to \operatorname{Hom}(H_q(X; Z), Z_{r_k}) \to 0.$$

Let $\bar{a}_k \in H^n(X; Z_{r_k})$ such that $\langle \bar{a}_k, a_k \rangle = 1 \in Z_{r_k}$, and $\langle \bar{a}_k, a_i \rangle = 0$ if $i \neq k$. Then

$$\begin{aligned} \langle \bar{a}_k, h(\alpha) \rangle &= \langle \bar{a}_k, t_1 a_1 + \cdots + t_m a_m \rangle \\ &= t_k \langle \bar{a}_k, a_k \rangle = t_k \neq 0 \in Z_{r_k}. \end{aligned}$$

So, since $(1)\lambda = h(\alpha)$, $t_k = \langle \bar{a}_k, h(\alpha) \rangle = \langle \bar{a}_k, (1)\lambda \rangle = \langle \bar{a}_k \bar{\lambda}, 1 \rangle$. So $\bar{a}_k \bar{\lambda} = t_k 1 \in H^0(X; Z_{r_k})$.

Since X is a suspension, $\bar{a}_k \cup \bar{b}_m = 0$. So

$$\begin{aligned} 0 = \phi^*(\bar{a}_k \cup \bar{b}_m) &= \phi^*(\bar{a}_k) \cup \phi^*(\bar{b}_m) \\ &= (\bar{a}_k \otimes 1 + t_k 1 \otimes \bar{\lambda}) \cup (\bar{b}_m \otimes 1) \\ &= t_k \bar{b}_m \otimes \bar{\lambda} = 0. \end{aligned}$$

So $t_k \bar{b}_m = 0$. But $t_k \neq 0$ and $t_k < r_k$, so that $t_k \bar{b}_m \neq 0$ since $\bar{b}_m$ has order $r_m \geqq r_k > t_k$. This contradiction shows that $h(\alpha)$ must have infinite order.

If $h(\alpha)$ has even dimension n, it is easy to see that the dual element $\bar{\beta} \in H^n(X; Z)$ has nonzero powers. See the previous theorem. This cannot happen in a suspension, so n must be odd.

Now we shall show that X is a rational cohomology n-sphere. Let $\bar{\beta} \in H^n(X; R)$ be the dual of $h(\alpha)$ in $H_n(X; R)$. Then $\bar{\beta}\bar{\lambda} = k1$ for some rational k. Suppose that $\bar{z} \in H^r(X; R)$, $r > 0$ is an element not a multiple of $\bar{\beta}$. Then $\bar{\beta} \cup \bar{z} = 0$. So

$$\begin{aligned} 0 = \phi^*(\bar{\beta} \cup \bar{z}) &= \phi^*(\bar{\beta}) \cup \phi^*(\bar{z}) \\ &= (\bar{\beta} \otimes 1 + k1 \otimes \bar{\lambda}) \cup (\bar{z} \otimes 1 + \bar{z}\bar{\lambda} \otimes \bar{\lambda}) \\ &= (\bar{\beta} \cup \bar{z}) \otimes 1 + (\bar{\beta} \cup \bar{z}) \otimes \bar{\lambda} + (-1)^{\dim \bar{z}}(k\bar{z}) \otimes \bar{\lambda} \end{aligned}$$

Hence $\bar{\beta} \cup \bar{z}\bar{\lambda} = -(-1)^{\dim \bar{z}}(k\bar{z})$. If $\bar{z}\bar{\lambda} \neq 0$ and $\dim \bar{z}\bar{\lambda} > 0$, then $\bar{\beta} \cup \bar{z}\bar{\lambda} = 0 = \pm kz$. Hence $\bar{z} = 0$. If $z\bar{\lambda} = 0$, then $\bar{\beta} \cup \bar{z}\bar{\lambda} = \bar{\beta} \cup 0 = 0$ implying $z = 0$. If $\bar{z}$ has dimension n and $\bar{z}$ is linearly independent of $\bar{\beta}$, then there exists $\bar{z}' \neq 0$ such that $\bar{z}'\bar{\lambda} = 0$. Thus $\bar{z}' = 0$. This contradiction establishes the theorem.

COROLLARY 5-3. *Let $\alpha \in G_n(X)$. If $h(\alpha) \in H_n(X; Z)$ is a generator for $H_n(X; Z)$, then X is homotopically equivalent to S^n.*

Proof. Suppose $h(\alpha)$ is not divisible in $H_n(X;Z)$. Let $\bar{\beta} \in H^n(X;Z)$ be the dual to $h(\alpha)$. This exists since $h(\alpha)$ is not a torsion element. Now $\bar{\beta}$ is not divisible. Therefore $\bar{\beta} \otimes Z_p$, for any prime p is not zero. We shall let $\bar{\beta}$ stand for $\mu(\bar{\beta} \otimes 1_p) \in H^n(X;Z_p)$. Now $\bar{\beta}\bar{\lambda} = 1 \in H^0(X;Z_p)$. If $\bar{v} \in H^r(X;Z_p)$, $r > 0$ then $\bar{\beta} \cup \bar{v} = 0$. But this implies, by the standard argument, that $-\bar{v}\bar{\lambda} \cup \bar{\beta} = \bar{v}$. This can only be true if $\bar{v} = 0$ or $\bar{v}\bar{\lambda} = k1$ for some $k \in Z_p$. This implies that $H^n(X;Z_p) \cong Z_p$ and $H^r(X;Z_p) = 0$ if $r \neq 0$ or n. This is true for all primes p, so it must be true for $H^*(X;Z)$. That is, X has the homology of S^n.

Since $\phi \mid S^n \colon S^n \to X$ sends the fundamental class of $H_n(S^n;Z)$ onto the generator $h(\alpha)$ of $H_n(X;Z)$, $\phi \mid S^n$ induces an isomorphism on the homology groups. Since S^n is simply connected for $n > 1$, we are finished. If $n = 1$, then $X = \Sigma Y$ and Y is a set of contractible path components. Thus ΣY is homotopically equivalent to a one dimensional graph, so if $H_1(\Sigma Y, Z) \cong Z$, ΣY is homotopy equivalent to S^1.

THEOREM 5-4. *$G_n(S^n) = 0$ if n is even*

$$= 2Z \subseteq Z = \pi_n(S^n) \text{ if } n \text{ odd and } n \neq 1, 3, 7$$

$$G^n(S^n) = Z = \pi_n(S^n) \text{ if } n = 1, 3, 7.$$

Proof. If n is even, then $G_n(S^n) = 0$ by Theorem 5-12 and the fact that $\ker h_\infty = 0 \in \pi_n(S^n)$.

Let n be odd. Then there exists a fibration $S^n \to V \xrightarrow{\omega} S^n$ where V is the Stiefel manifold $V_{n+2,2}$ of all unit tangent vectors on S^{n+1}. Is is known, see page 323 of Hu [13], that the image of the boundary homomorphism $d \colon \pi_{n+1}(S^{n+1}) \to \pi_n(S^n)$ is a subgroup of index two in $\pi_n(S^n)$. Thus $2Z \subseteq G_n(S^n) \subseteq \pi_n(S^n) = Z$ if n is odd. If $G_n(S^n) = \pi_n(S^n)$, there would exist a $(1,1)$ map $S^n \times S^n \to S^n$, and so S^n would be an H-space. Thus $G_n(S^n) = \pi_n(S^n)$ if and only if $n = 1$, 3 or 7. (Adams [1]).

COROLLARY 5-5. *Suppose ΣX has homology of finite type and suppose there is an $\alpha \in G_n(\Sigma X)$ such that $h(\alpha)$ is a generator of $H_n(\Sigma X;Z)$. Then $n = 1$, 3 or 7 and ΣX is homotopy equivalent to S^n.*

Proof. By Corollary 5-3 and Corollary 514, we are finished.

6. Coverings maps and $G_n(X)$. The purpose of this section is to investigate the relationship between $G_n(X)$ and the evaluation subgroup $G_n(\tilde{X})$ for the n-connective covering space $\tilde{X}$. We begin by studying covering spaces in the first part of the section, and then we consider fibrations with fibre $K(\pi, n)$. We obtain a general result, Theorem 6-3, which allows us to generalize our results about covering spaces to n-connective covering spaces.

In [10], we considered the case where $\tilde{X}$ is the universal covering of X and we classified $G_1(X)$ in terms of the Deck Transformations of $\tilde{X}$. We shall begin this investigation by considering $G_1(\tilde{X})$, where $\tilde{X}$ is any covering space of X.

THEOREM 6-1. *Let $p\colon \tilde{X} \to X$ be a covering of X. If $p_*(\alpha) \in G_1(X)$, then $\alpha \in G_1(\tilde{X})$.*

Proof. Since $p_*(\alpha) \in G_1(X, x_0)$, there is a map $\phi\colon X \times S^1 \to X$ affiliated to $p_*(\alpha)$. This map ϕ allows us to define a homotopy $h_t\colon X \to X$ such that $h_0 = h_1 = 1_X$ and the closed path $\sigma\colon t \to h_t(x_0)$ represents $p_*(\alpha)$. By the covering homotopy property, we can find a homotopy $\tilde{h}_t\colon \tilde{X} \to \tilde{X}$ which covers h_t such that $\tilde{h}_0 = 1_{\tilde{X}}$. Then $\tilde{\sigma}\colon t \to \tilde{h}_t(\tilde{x}_0)$ lifts the path σ. Here $\tilde{x}_0$ is a base point of $\tilde{X}$ such that $p(\tilde{x}_0) = x_0$. Since σ represents $p_*(\alpha)$, σ must be covered by a closed path in $\tilde{X}$ which represents α. This path must be $\tilde{\sigma}$ since $p\tilde{\sigma} = \sigma$. Thus $\tilde{\sigma}(0) = \tilde{\sigma}(1) = \tilde{x}_0$.

Now $\tilde{h}_1\colon \tilde{X} \to \tilde{X}$ is a covering transformation (i.e. covers 1_X). Also $\tilde{h}_1(\tilde{x}_0) = \sigma(1) = \tilde{x}_0$. The only covering transformation which has a fixed point is the identity, $1_{\tilde{X}}$. Thus $\tilde{h}_0 = \tilde{h}_1 = 1_{\tilde{X}}$ and σ represents α, so using $\tilde{h}_t$ we may construct an affiliated map to α. Hence $\alpha \in G_1(\tilde{X}, \tilde{x}_0)$.

THEOREM 6-2. *Let $p\colon \tilde{X} \to X$ be a covering map. If $n > 1$, then $p_*^{-1}(G_n(X)) \subseteq G_n(\tilde{X})$. In other words, if we identify $\pi_n(X)$ with $\pi_n(\tilde{X})$ under the isomorphism p_*, then $G_n(\tilde{X}) \supseteq G_n(X)$.*

Proof. If $\alpha \in G_n(X)$, there is a map affiliated with α. Call it $\phi\colon X \times S^n \to X$. We may lift ϕ to $\tilde{\phi}\colon \tilde{X} \times S^n \to \tilde{X}$ so that the following diagram commutes.

$$\begin{tikzcd}
\tilde{X} \times S^n \arrow[r, "\tilde{\phi}"] \arrow[d, "p \times 1"] & \tilde{X} \arrow[d, "p"] \\
X \times S^n \arrow[r, "\phi"'] & X
\end{tikzcd}$$

The existence of $\tilde{\phi}$ is guaranteed by a well known result about covering spaces, see Theorem 16.2, [13].

Now $\tilde{\phi} \mid \tilde{X}$ is the identity on $\tilde{X}$ and $\tilde{\phi} \mid S^n$ represents $p_*^{-1}(\alpha)$. Thus $p_*^{-1}(\alpha) \in G_n(\tilde{X})$.

Suppose we have a principal fibration $p\colon E \to B$ with fibre a $K(\pi, r)$. Here $r > 0$. Suppose there exists a map $\phi\colon B \times S^n \to B$ such that $\phi \mid B$ is the identity on B. Under what conditions does there exist a map $\tilde{\phi}\colon E \times S^n \to B$ such that $\phi \mid E$ is the identity on E and the following diagram commutes

$$\begin{tikzcd} E \times S^n \arrow[r, "\tilde{\phi}"] \arrow[d, "p \times 1"] & E \arrow[d, "p"] \\ B \times S^n \arrow[r, "\phi"] & B \end{tikzcd} \qquad ?$$

The map $\phi\colon B \times S^n \to B$ induces structure on the cohomology of B. We shall use the notation of § 5.

Now there exists a map $f\colon B \to K(\pi, r+1)$ such that the induced fibration of f from the fibration $PK \to K(\pi, r+1)$, where PK is the space of paths over $K(\pi, r+1)$, is fibre homotopy equivalent to $p\colon E \to B$. Let $l \in H^{r+1}(K(\pi, r+1); \pi)$ be a characteristic element for $K(\pi, r+1)$. Define $\mu = f^*(l) \in H^{r+1}(B; \pi)$.

THEOREM 6-3. *With the above notation, there exists a map $\tilde{\phi}\colon E \times S^n \to E$ such that $\tilde{\phi} \mid E$ is the identity on E and*

$$\begin{tikzcd} E \times S^n \arrow[r, "\tilde{\phi}"] \arrow[d, "p \times 1"] & E \arrow[d, "p"] \\ B \times S^n \arrow[r, "\phi"] & B \end{tikzcd}$$

commutes if and anly if $\mu\bar{\lambda} = 0$.

Proof. Let $c\colon (D^n, S^{n-1}) \to (S^n, s_0)$ be the usual quotient map. Then we have the diagram

$$\begin{tikzcd} & & E \arrow[r, "\tilde{f}"] \arrow[d] & PK \arrow[d] \\ B \times D^n \arrow[r, "1 \times c"] & B \times S^n \arrow[r, "\phi"] & B \arrow[r, "f"] & K(\pi, r+1). \end{tikzcd}$$

Here $\tilde{f}$ is a fibre map.

We may lift $\phi \circ (1 \times c)$ to a map Φ by the homotopy covering property such that the following diagram commutes:

$$\begin{tikzcd}
E \times D^n \arrow[rr, "\Phi"] \arrow[d, "p \times 1"'] & & E \arrow[d, "p"] \\
B \times D^n \arrow[r, "1 \times c"] & B \times S^n \arrow[r, "\phi"] & B
\end{tikzcd}$$

Consider $\Phi \mid E \times S^{n-1}$. Call this map Φ'. Then we have

$$\begin{tikzcd}
E \times S^{n-1} \arrow[r, "\Phi'"] \arrow[d, "p \times *"] & E \arrow[d, "p"] \\
B \arrow[r, "1_B"'] & B
\end{tikzcd}$$

as a commutative diagram, and $\Phi' \colon E \times s \to E$ is a fibre homotopy equivalence for each $s \in S^{n-1}$. Here $* \colon S^{n-1} \to B$ is a constant map.

Now note that we may have a commutative diagram

$$\begin{tikzcd}
E \times S^{n} \arrow[r, "\tilde{\phi}"] \arrow[d, "p \times 1"] & E \arrow[d, "p"] \\
B \times S^n \arrow[r, "\phi"] & B
\end{tikzcd}$$

if and only if it is possible to find a homotopy $\Phi'_t \colon E \times S^{n-1} \to E$ such that

$$\begin{tikzcd}
E \times S^{n-1} \arrow[r, "\Phi'_t"] \arrow[d, "p \times *"] & E \arrow[d, "p"] \\
B \arrow[r, "1_B"] & B
\end{tikzcd}$$

commutes for all $t \in I$, and such that $\Phi'_0 = \Phi'$ and $\Phi'_1(e,s) = e$ for all $s \in S^{n-1}$. For in this case, we may define $S^n = D_-^n \cup D_+^n$ where $D_+^n \cap D_-^n = S^{n-1}$, and then define $\tilde{\phi} \colon E \times S^n \to E$ by $\tilde{\phi}(e,s) = \Phi(e,s)$ if $s \in D_-^n$ and $\tilde{\phi}(e,s) = \Phi'_t(e,s')$ where (s',t) represents the point $s \in D_+^n$ (we represent D_+^n as the cone over S^{n-1}). This definition of $\tilde{\phi}$ covers a map $h \colon B \times S^n \to B$ defined by $h(b,s) = \phi \circ (1 \times c)(b,s)$ for $s \in D_-^n$ and $h(b,s) = b$ for $s \in D_+^n$. Now h is clearly homotopic to ϕ, so by the covering homotopy property we may alter ϕ so that it lifts $\tilde{\phi}$.

In [11], the author studied just such a situation. Letting $L^{**}(E,E)$ be the space of fibre homotopy equivalences from E to E, he denoted the group of homotopy classes of such mappings $\Phi': E \times S^{n-1} \to E$ by $Q_{n-1}(L^{**}(E,E))$. (The homotopies Φ'_t must cover $1: B \to B$ for all t).

Now there exists a classifying fibration $p_\infty: E_\infty \to B_\infty$ for fibrations with fibre $K(\pi, r)$. See [7], p. 16.8 or [2]. Suppose that $k: K(\pi, r+1) \to B_\infty$ is a classifying map for the fibration $PK \to K(\pi, r+1)$. Then by Lemma 4, p. 48 [11], $Q_{n-1}(L^{**}(E,E)) \cong \pi_n(L(B, B_\infty); k \circ f)$. In addition, the isomorphism is given by the correspondence which relates the homotopy class of $\Phi': E \times S^{n-1} \to E$ to the homotopy class of $k \circ f \circ \phi: B \times S^n \to B_\infty$. We can obtain the required homotopy Φ'_t if and only if $k \circ f \circ \phi$ is homotopic to $k \circ f \circ p_1$ where $p_1: B \times S^n \to B$ is projection into the first factor. Now $f \circ \phi \sim f \circ p_1$ if $\phi^* \circ f^*(l) = p_1^* \circ f^*(l)$. Since

$$\phi^* f^*(l) = \phi^*(\mu) = \mu \otimes 1 + \mu\bar{\lambda} \otimes \bar{\lambda}$$

and $p_1^* f^*(l) = p_1^*(\mu) = \mu \otimes 1$, we have the required homotopy if $\mu\bar{\lambda} = 0$.

To show that $\mu\bar{\lambda} = 0$ implies $k \circ f \circ \phi \sim k \circ f \circ p_1$, we must prove the following lemma.

LEMMA 6-4. *$k: K(\pi, r+1) \to B_\infty$ is homotopic to $h: K(\pi, r+1) \to B_\infty$, where h is the universal covering map from $K(\pi, r+1) = \tilde{B}_\infty$ to B_∞.*

Proof. Let B_∞ be the classifying space of fibrations with fibres $K(\pi, r)$. Let $\tilde{B}_\infty$ be the covering space of B_∞. Then we get the following commutative diagram:

$$\begin{tikzcd}
K(\pi, r) \arrow[r] \arrow[d] & K(\pi, r) \arrow[r] \arrow[d] & K(\pi, r) \arrow[d] \\
PK \arrow[r] \arrow[d] & h^*(E_\infty) \arrow[r] \arrow[d] & E_\infty \arrow[d, "p_\infty"] \\
K(\pi, r+1) \arrow[r, "\tilde{k}"] & \tilde{B}_\infty \arrow[r, "h"] & B_\infty
\end{tikzcd}$$

Here h is the covering map and $h^*(E_\infty)$ is the induced total space. Since $r+1 > 1$, $K(\pi, r+1)$ is simply connected and so $k: K(\pi, r+1) \to \tilde{B}_\infty$ can be lifted to a map $\tilde{k}: K(\pi, r+1) \to \tilde{B}_\infty$. The top horizontal arrows are homotopy equivalences in the above diagram.

This diagram gives rise to a commutative diagram of homotopy groups using the fibre homotopy exact sequence

$$\begin{tikzcd}
\pi_{r+1}(K(\pi, r+1)) \arrow[r, "\tilde{k}_*"] \arrow[d, "\cong"] & \pi_{r+1}(B_\infty) \arrow[r, "h_*", "\cong"'] \arrow[d, "\tilde{d}_\infty"] & \pi_{r+1}(B_\infty) \arrow[d, "d_\infty"] \\
\pi_r(K(\pi, r)) \arrow[r, "\cong"'] & \pi_r(K(\pi, r)) \arrow[r, "\cong"'] & \pi_r(K(\pi, r))
\end{tikzcd}$$

Now, by Theorem 5.1 in [2] we see that

$$\omega_*: \pi_n(L(K(\pi,n),K(\pi,n));1_K) \cong \pi_n(K(\pi,n)) \text{ for } n \geqq 2.$$

If $n = 1$, the result still holds by applying Theorem II.1 of [10]. The fact that ω_* is an isomorphism implies that d_∞ is an isomorphism, see [11]. This implies that $\tilde{d}_\infty$ is an isomorphism and thus that

$$\tilde{k}_*: \pi_{r+1}(K(\pi, r+1) \cong \pi_{r+1}(\tilde{B}_\infty)$$

is an isomorphism.

In fact $\tilde{B}_\infty$ is a $K(\pi, r+1)$, because $\pi_i(B_\infty) \cong \pi_{i-1}(L(K,K),1_K)$ where $K = K(\pi, r)$. By Thom, [18], $\pi_0(L,1_K) \cong \mathcal{E}(K)$, the group of homotopy equivalences from K to K and $\pi_r(L,1_K) \cong \pi$, the other homotopy groups being zero. Thus $\pi_i(\tilde{B}_\infty) = 0$ unless $i = r+1$, then $\pi_{r+1}(\tilde{B}_\infty) = \pi$. This establishes the lemma and hence the theorem.

As a consequence of Theorem 6-3, we have a generalization of Theorem 6-2.

COROLLARY 6-5. *Suppose X is $(n-1)$-connected. Let $\tilde{X}$ be an n-connective covering of X. Then there is a fibration $p: \tilde{X} \to X$ which induces isomorphisms $p_*: \pi_i(\tilde{X}) \to \pi_i(X)$ for all $i > n$. Identifying $\pi_i(\tilde{X})$ to $\pi_i(X)$ by p_*, we see that $G_i(X) \subseteq G_i(\tilde{X})$.*

Proof. The fibration $p: \tilde{X} \to X$ has fibre a $K(\pi, n-1)$ where $\pi = \pi_n(X)$. To show that $G_i(X) \subseteq G_i(\tilde{X})$, $i > n$, we need only show that any map $\phi: X \times S^i \to X$ affiliated with an element in $G_i(X)$ gives rise to a commutative diagram

$$\begin{array}{ccc} \tilde{X} \times S^i & \xrightarrow{\tilde{\phi}} & \tilde{X} \\ \Big\downarrow {\scriptstyle p\times 1} & & \Big\downarrow {\scriptstyle p} \\ X \times S^i & \xrightarrow[\phi]{} & X \end{array}$$

such that $\tilde{\phi} \mid \tilde{X}$ is the identity on $\tilde{X}$. By Theorem 6-3, this occurs if and only if $\mu\lambda = 0$ where $\mu \in H^n(X;\pi)$ which defines the fibration $p: \tilde{X} \to X$.

Since λ lowers dimension by i, and since $i > n$, $u\lambda = 0$.

7. Applications. We shall combine the results of the preceding sections to obtain some interesting theorems.

THEOREM 7-1. *Let X have only a finite number of nonzero rational homology groups, then $G_2(X) \subseteq$ torsion subgroup of $\pi_2(X)$.*

Proof. Let $\tilde{X}$ be the universal covering space of X. We may identify $\pi_2(X)$ with $\pi_2(\tilde{X})$ by the isomorphism p_*. Then, by Theorem 6-2, we have $G_2(\tilde{X}) \supseteq G_2(X)$. Since $\tilde{X}$ has only a finite number of nonzero rational homology groups, by Theorem 5-1, $G_2(\tilde{X}) \subseteq \ker h_\infty$. Since $\tilde{X}$ is simply connected, the Hurewicz Theorem tells us that $h\colon \pi_2(\tilde{X}) \cong H_2(\tilde{X})$. Thus the kernel of h_∞ must be those elements of $\pi_2(\tilde{X})$ which are killed by tensoring with the rationals, that is, the torsion subgroup of $\pi_2(\tilde{X})$. Thus $G_2(X) \subseteq G_2(\tilde{X}) \subseteq$ torsion subgroup of $\pi_2(\tilde{X}) =$ torsion subgroup of $\pi_2(X)$.

Suppose π is a finitely generated abelian group. Then $\pi = F \oplus T$ where T is the torsion subgroup of π and F is a free group. We may regard

$$K(\pi, n) = K(F, n) \times K(T, n).$$

Now $H^*(K(T,n); R)$, where R is the rational numbers, is trivial except for H^0. Also $H^*(K(Z,n); R))$ is isomorphic over R to an exterior algebra generated by an element of degree n if n is odd and is isomorphic to a polynomial algebra generated by an element of degree n if n is even. Thus, when n is odd, since $F = Z \oplus \cdots \oplus Z$ for a finite number of summands,

$$H^*(K(\pi, n); R) = H^*(K(Z,n); R) \otimes \cdots \otimes H^*(K(Z,n); R),$$

has only a finite number of nonzero rational homology groups. We use the above facts in the proof of the following theorem:

THEOREM 7-2. *Suppose X is a CW complex with a finite number of nonzero rational homology groups. Let $\pi_i(X)$ be finitely generated and suppose that $\pi_i(X)$ is a finite group for all odd i, $1 < i < N$. Then $G_i(X)$ is a finite subgroup for all integers i such that $1 < i < N$.*

Proof. Let $\tilde{X}$ be the universal covering space of X. Then $\tilde{X}$ satisfies the hypothesis of the theorem. Then we know, Theorem 7-1, that $G_2(X) \subseteq G_2(\tilde{X}) \subseteq$ torsion subgroup of $\pi_2(X)$ which is a finite group since $\pi_2(X)$ is finitely generated. Let $\tilde{X}_2$ be the 2-connected covering of $\tilde{X}$. Then we have the fibration $K(\pi_2(X), 1) \to \tilde{X}_2 \to \tilde{X}$. Since both the base and the fibre have only a finite number of nonzero rational homology groups, it follows from the Serre spectral sequence of a fibration that $\tilde{X}_2$ has only a finite number of nonzero rational homology groups. So $\tilde{X}_2$ satisfies the hypothesis of the theorem. Now $G_3(X)$ is finite if $\pi_3(X)$ is finite.

Consider $\tilde{X}_3$, the 3-connective covering of $\tilde{X}_2$. Then we have the fibration

$$K(\pi_3(X), 2) \to \tilde{X}_3 \to \tilde{X}_2.$$

Since $\pi_3(X)$ is finite, $K(\pi_3(X), 2)$ has trivial positive dimensional homology groups. So, by the Serre spectral sequence, the rational homology groups of $\tilde{X}_3$ are isomorphic to those of $\tilde{X}_3$ and thus there are only a finite number of nonzero groups. Hence $\tilde{X}_3$ satisfies the hypotheses of the theorem.

Now $\tilde{X}_3$ is 3-connected, so by the Hurewicz theorem, the kernel of the Hurewicz homomorphism $h\colon \pi_4(\tilde{X}_3) \to H_4(\tilde{X}_3)$ is zero. Hence the kernel of h_∞ is the torsion subgroup of $\pi_4(\tilde{X}) \cong \pi_4(X)$. Since $\tilde{X}_3$ has only a finite number of nonzero homology groups, $G_4(\tilde{X}_3) \subseteq \ker h_\infty =$ torsion subgroup of $\pi_4(\tilde{X}_3)$. Since $G_4(X) \subseteq G_4(\tilde{X}_3)$, we see that $G_4(X)$ is contained in the torsion subgroup of $\pi_4(X)$ and hence must be finite.

We repeat the arguments for $\tilde{X}_2$ and $\tilde{X}_3$ for $\tilde{X}_{2i}$ and $\tilde{X}_{2i+1}$ in an alternating pattern until $i = N$.

This last theorem is interesting, for we apply results which depend upon a finite number of nonzero homology groups far beyond the largest dimension of the nonzero homology groups.

As another application of our results, we shall investigate the concept of a G-space.

Definition. We call a space for which $G_n(X) = \pi_n(X)$ for all n a G-space. Any H-space is a G-space. For finitely generated homology, $H_*(X)$, G-spaces and H-spaces share a striking property.

THEOREM 7-3. *Let X be a connected G-space such that $H_*(X;Z)$ is finitely generated. Then either $\chi(X) = 0$ or X is contractible.*

Proof. First we shall show that X is either simply connected or $\chi(X) = 0$. By the definition of G-space, $G_1(X) = \pi_1(X)$. Thus by Corollary 2-4, $\pi_1(X)$ is abelian. Hence the Hurewicz map h is an isomorphism. Suppose $\pi_1(X) \neq 0$. Then there is a generator $\alpha \in \pi_1(X)$. Let p be a prime which divides the order of α (possibly ∞). Then $h(\alpha) \otimes Z_p \neq 0$. So by Theorem 4.1, $\chi(X) = 0$, since $h_p(\alpha) \neq 0$ and $\alpha \in G_1(X)$.

Now suppose that $\chi(X) \neq 0$. Then X is simply connected. Assume also that $\chi(X) \neq 1$. Then for some $n > 0$, $H_n(X;Z)$ has a torsion free element. Assume that n is the smallest such integer. Let $\mathcal{T}$ be the Serre class of all torsion groups. See [13], for example. Then, since X is simply connected, we may apply the mod $\mathcal{T}$ Hurewicz theorem, see p. 305 [13], which tells us that $h\colon \pi_n(X) \to H_n(X;Z)$ is a $\mathcal{T}$-isomorphism. Thus there is some torsion free $\beta \in H_n(X;Z)$ such that $h(\alpha) = \beta$ for some $\alpha \in \pi_n(X) = G_n(X)$. Hence $\alpha \notin \ker h_\infty$ and $\alpha \in G_n(X)$, so by Theorem 5-1, this contradicts the fact that $H_*(X;Z)$ is finitely generated if n is even. If n is odd, then $\chi(X) = 0$ by Theorem 4-1, so we have a contradiction.

Thus if $\chi(X) \neq 0$, then $\chi(X) = 1$. In this case we use the Hurewicz theorem and Corollary 4-6 or Theorem 4-1. Suppose there is a $n > 0$ such that $H_n(X;Z) \neq 0$. Then let n be the smallest such integer. Since $\pi_1(X) = 0$, the Hurewicz theorem tells us that $h: \pi_n(X) \to H_n(X;Z)$ is an isomorphism. So let β be a generator of $H_n(X;Z)$. Then there exists an $\alpha \in G_n(X) = \pi_n(X)$ such that $h(\alpha) = \beta$. Suppose p divides the order of β. Then $h_p(\alpha) \neq 0$. If n is even, we use Corollary 4-6 to produce a contradiction. If n is odd, we use Theorem 4-1 to show that $\chi(X) = 0$. Thus if $\chi(X) = 1$, $H_n(X;Z) = 0$ must be zero for all $n > 0$. Since $\pi_1(X) = 0$, we see that $\pi_n(X) = 0$ for all n and so X is contractible.

Using Corollary 2-7 and various of our above results, we may obtain results of the following type.

THEOREM 7-4. *Let $p: E \to S^n$ be a fibration with fibre F a CW complex with finitely generated homology. There is a cross-section to this fibration if one of the following conditions hold: ($n > 2$ except where specified)*

1) *$n = 2$ and $\pi_1(F)$ has trivial center.*

2) *$n = 3$ and $\pi_2(F)$ has no torsion.*

3) *F is $n-2$ connected, $\chi(F) = 1$, and the order of the torsion subgroup of $\pi_{n-1}(F)$ is square free.*

4) *F is $n-2$ connected, $\chi(F) \neq 0$, and the order of the torsion subgroup of $\pi_{n-1}(F)$ is square free and n is even.*

5) *F is a $n-2$ connected suspension not an odd dimensional rational homology sphere.*

Proof. A cross-section exists if $G_{n-1}(F) = 0$ by Corollary 2-7. Now $G_{n-1}(F) = 0$ in the above five cases by Theorem 2-4, Theorem 7-1, Theorem 4-6, Theorem 4-1, and Theorem 5-2 respectively. For 3), 4) and 5) we also need the Hurewicz isomorphism theorem.

8. Questions. Theorem 3-1, which says that $G_1(X)$ is trivial if X is a compact polyhedron and $\chi(X) \neq 0$, is proved using the Nielsen-Wecken fixed point theory. On the other hand, Theorem 4-1 tells us that $G_1(X) \subseteq \ker h_p$ for all primes p if $\chi(X) \neq 0$ and X has finitely generated homology. The proof is an application of homology theory, and thus generalizes to dimensions greater than $n = 1$, but the conclusion is weaker.

It would be interesting to find a proof of Theorem 3-1 which generalizes to higher dimension. In particular, does Theorem 4-1 imply Theorem 3-1?

Let X have finitely generated homology. If X is $n-1$ connected, where

n is odd, and if $\chi(X) \neq 0$, then $G_n(X)$ is trivial if every element of $H_n(X)$ has order a number whose prime factorization contains a prime of power one.

This theorem would generalize Theorem 3-1 if we could remove the condition on the order of elements of $H_n(X)$. This can be done if we can improve Theorem 4-1 from $G_n(X) \subseteq \ker h_p$ for all primes p to $G_n(X) \subseteq \ker h$.

For suitable conditions on X, Theorems 5-1 tells us that $G_{2n}(X) \subseteq \ker h_\infty$. One may ask if this theorem can be improved to $G_{2n}(X) \subseteq \ker h_p$ for primes p. This would put the conclusion of Theorem 5-1 in agreement with the conclusion of Theorem 4-1.

It is known that $G_{2n}(X) \subseteq \ker h_p$ if $H_*(X)$ is finitely generated and $\chi(X) = 1$ (Corollary 4-6), or if X is suspension (Theorem 5-2), or if X is an H-space with finitely generated homology. This last result follows from a theorem of W. Browder [4] which states that

$$h_p\colon \pi_{2n}(X) \to H_{2n}(X; Z_p)$$

is the zero map. Since $G_n(X) = \pi_n(X)$ for all n, $G_n(X) \subseteq \ker h_p$.

We may also ask if $G_{2n}(X) \subseteq \ker h$ when $H_*(X)$ is finitely generated. This conjecture implies that $G_2(X) = 0$ if X has finitely generated homology. This fact, then, would imply that every fibre space over S^3, with fibre F such that $H_*(F)$ is finitely generated, admits a cross-section.

Finally, it would be interesting to know if there exists a G-space which is not an H-space.

Added in proof. Jerrold Siegel has a finite dimensional example, the author and H. B. Haslam have examples with a finite number of non-zero homotopy groups.

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