A gravitational lens need not produce an odd number of images

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Given any space–time M without singularities and any event O, there is a natural continuous mapping f of a two-dimensional sphere into any spacelike slice T not containing O. The set of future null geodesics (or the set of past null geodesics) form a two-sphere S^2 and the map f sends a point in S^2 to the point in T which is the intersection of the corresponding geodesic with T. Considering the f for each point of a world-line W gives us a map $F: S^2 \times W \rightarrow T$. The local degree of F at a regular value y in T has the same parity as the number of null geodesics from W to y.

I. INTRODUCTION

Since 1979 astronomers have been looking for an odd number of images in gravitational lensing events. There have been many discoveries since the first event in 1979. In most cases *only* an even number of discrete images have been found. Some of the topological arguments for an odd number of images are very persuasive, even though they are based on a Euclidean space–time.

In 1980 Dyer and Roeder[1] predicted an odd number of images for a spherical symmetric transparent lens (i.e., Galaxy). In 1981 Burke[2] claimed that there must be an odd number of images for any bounded transparent lens subject to an assumption that the bending of light rays decreases as the light rays are far from the lens. The argument constructed a vector field on the plane of the lens and showed the index had to be 1. So then, assuming the local index of each zero was ± 1, the number of zeros had to be odd to add up to the global index of 1. Each zero corresponds to light rays.

In 1985 McKenzie[3] wrote down an argument using the degree of a map between two two-dimensional spheres which asserted that there were an odd number of images. This argument needed no assumptions on the amount of bending and obviously improved Burke's approach. This argument was widely known among astronomers and is very convincing. However it is done in three-space and not in four-dimensional space–time. McKenzie notes this and then provides an argument using Morse theory on four-dimensional space–time, applying correctly Karen Uhlenbeck's version of Morse theory for Lorentzian manifolds.[4] It is widely believed today that the necessity of an odd number of images has been precisely established and that the contradictory evidence is a result of difficulties of finding the third image,[5] although on page 176 of Ref. 6 they state that McKenzie's conditions are physically obscure.

In this article we translate the degree argument directly into four-dimensional space–time and we give a necessary and sufficient condition for an odd number of images.

We give two examples of four-dimensional Lorentzian manifolds for which this condition is false: the first one because of the topology and the second one because of the geometry. Then we argue that the conditions under which McKenzie's Morse theory argument would apply are extremely restrictive.

II. GLOBAL LENSING IN LORENTZIAN SPACE–TIME

We reproduce the topological argument given by McKenzie on page 1592 of Ref. 3 which establishes the odd image result for Euclidean space. Then we try to reproduce the argument in Lorentzian space–time.

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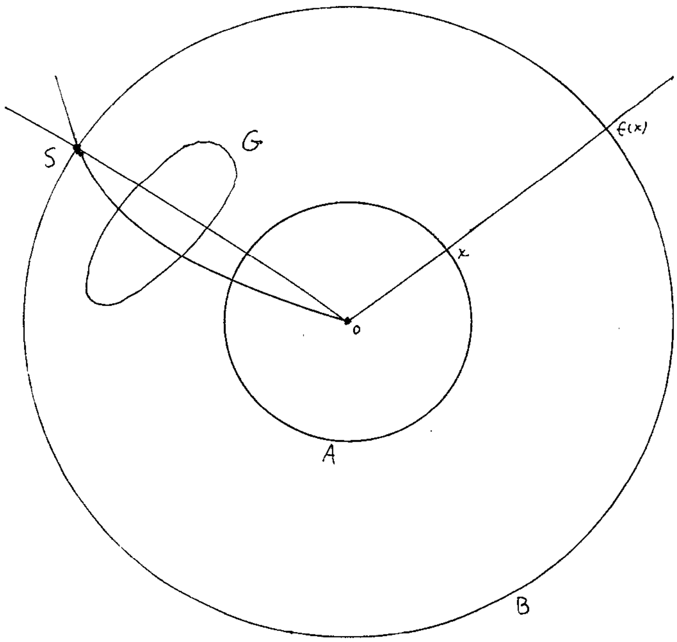

FIG. 1. A galaxy G is located somewhere between a light-source S and an observer O. Because of the gravitational field of the galaxy there may be more than one light ray from S to O. f maps the sphere A onto the sphere B. If x is on A then $f(x)$ is defined to be the point on B where the ray through O and x intersects B.

"There is a relatively simple demonstration of why there are an odd number of images. Although it seems to be well known among astronomers it does not appear to have been published before and so is given here. Consider the situation shown in Fig. 1. A light source is located at S and an observer at O. There is a transparent galaxy G somewhere between S and O. A map f from the small sphere A to the sphere B is defined as follows. The map f maps a point x on A to the point on B where the light ray through O and x intersects B. The number of images of S seen by O is the number of points on A mapped onto S.

Suppose g: $M \rightarrow N$ is a smooth map between manifolds of the same dimension and that M is compact. If y is a regular value of g then we define

$$\deg(g,y) = \sum_{x \in g^{-1}(y)} \operatorname{sgn} dg_x,$$

where $\operatorname{sgn} dg_x = +1(-1)$ if $dg_x : T_x(M) \rightarrow T_y(N)$ preserves (reverses) orientations. It turns out that $\deg(g,y)$ is the same for all regular y; it is called the degree of g and denoted $\deg(g)$.

In an actual physical situation it is reasonable to assume that there will be a point y on B such that $f^{-1}(y)$ is a single point, i.e., there is only one ray from O to y. Thus, $\deg(f) = 1$.

Let $n_+(n_{-1})$ be the number of points x in $f^{-1}(S)$ such that $\operatorname{sgn} df_x = +1(-1)$. Thus, $n_+(n_{-1})$ is the number of images of S, seen by O, which have the same (opposite) orientation as the source, and

$$n_+ - n_- = \deg(f,S) = \deg(f) = 1.$$

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Thus, if O sees $n=n_+-n_-$ images of S then $n=2n$, and so n is odd, and the demonstration is complete."

Now we consider this argument in Lorentzian space–time, M. Let y be the observer in a spacelike slice T and let the source follow the world-line W. Then there is an odd number of images at y if and only if the local degree of F at y is odd. If T is Euclidean three-space then the local degree is equal to the winding number of the image of the boundary of $S^2\times W$ with respect to y. In mild regimes where the pencils of geodesics intersect T in singular surfaces approximating spheres, this winding number is one.

III. TWO EXAMPLES

We give two examples of space–times which do not have the property that pencils of null geodesics intersect spacelike slices in two-spheres. Many more examples can be constructed using Barrett O'Neill's book,[7] Corollary 57 on page 89 and warped products on pages 207–209.

(a) Let $M=S^1\times S^1\times S^1\times\mathbb{R}$. The universal covering space is $\tilde{M}=\mathbb{R}^4$. Let $\tilde{M}$ be Minkowski space, so it has the Minkowski metric. It induces the same metric on M. The geodesics of $\tilde{M}$ are straight lines and their images are the geodesics of M. Pencils of null geodesics do not intersect spacelike slices in spheres in this M.

(b) Let $M=\mathbb{R}^4=\mathbb{R}^2\times\mathbb{R}^2$. Let the second $\mathbb{R}^2$ have the Minkowski metric. We will put a Riemannian metric on the first $\mathbb{R}^2$ and then we take the product metric. We note that a geodesic of the first $\mathbb{R}^2$ factor coupled with a timelike line in the second factor is a null geodesic in M, (i.e., if α: $\mathbb{R}\to\mathbb{R}^2$ is a geodesic of the first factor and β: $\mathbb{R}\to\mathbb{R}^2$ is a timelike geodesic of the second factor with the same speed as α, then $\alpha\times\beta:\mathbb{R}\to\mathbb{R}^2$ is a null geodesic of M). So if we produce an $\mathbb{R}^2$ so that the exponential map of geodesics emanating from a point x carries some circle in the tangent plane at x into a set in $\mathbb{R}^2$ which is not a topological circle, then the pencil of null geodesics intersecting a spacelike slice in M is not a two-sphere.

One can visualize a metric on $\mathbb{R}^2$ by embedding it as a surface T in Euclidean three-space. The geodesics are characterized as those paths in T whose acceleration is orthogonal to the surface T. Now it is easy to construct examples with the desired property.

One that works is the following. Take an arc of a circle whose length exceeds a half circle. Extend the ends of this arc by the tangent lines at the ends of the arc. The lines intersect in a point A. Now take a small interval perpendicular to the plane in which the curve just constructed, γ, lies. Move this interval along γ so that it is perpendicular to the plane over the arc and so that it lies in the plane along most of the two extended lines including their intersection A. The interval should be twisted in moving from the ends of the arc so that the interval sweeps out a smooth surface with two boundary components. Then extend this "old fashioned men's collar" to a surface T in $\mathbb{R}^3$.

Let O be the midpoint of the circular arc on T. Then the geodesics of fixed length greater than OA on T near γ clearly do not end in a circle.

We can adjust this example so that the nonflat part of $T\times\mathbb{R}$ is bounded in any space-like slice $T\times\mathbb{R}\times s\subset\mathbb{R}^2\times\mathbb{R}\times\mathbb{R}=M$. The technique for the adjustment is the warped product construction, which can be found in Ref. 7.

IV. MORSE THEORY

McKenzie in Ref. 3 studies the odd image result by applying Uhlenbeck's version of Morse theory of Lorentzian manifolds.[4] The relevant theorems are Theorem 4 (which he calls the local theorem) and Theorem 5 (the global theorem). The global theorem is less relevant to the study of gravitational lensing than the local theorem according to McKenzie. This is the case both for practical considerations of how observations are made, and because the hypotheses of the global theorem do not hold in realistic space–time models.

The statement of the local theorem is difficult to understand since McKenzie does not make clear how the points q and r and set B are related to the history of the source T and the observer

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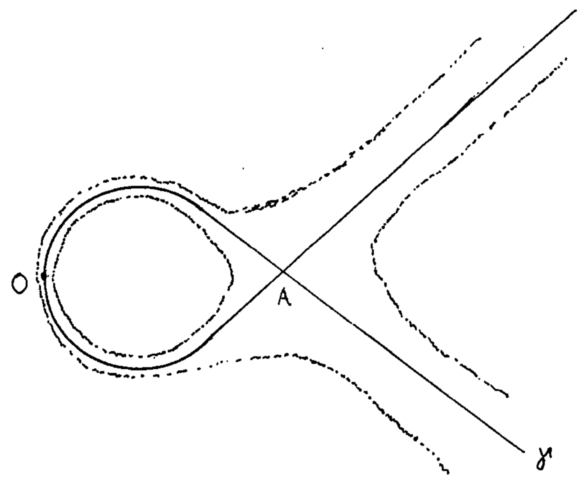

FIG. 2. Old fashioned men's collar with geodesic γ.

p. The most reasonable interpretation is that $\Omega(T,p)^c$ is a deformation retract of $\Omega(T,p)$ which he assumes is contractible. This is a wordy way of assuming that $\Omega(T,p)^c$ is contractible. Now as c varies, $\Omega(T,p)^c$ will not be contractible in general since every time c passes through a critical value of T, the topology of $\Omega(T,p)^c$ is altered by attaching a cell (which corresponds to a new geodesic from T to p). But it is impossible to attach only one cell to a contractible space and still have it be contractible. Thus for "most" c the hypothesis is not true unless there are pairs of geodesics from T to p for each critical value for c.

V. DISCUSSION

The argument in Sec. II has a flaw beyond its Euclidean setting. It depends on an unstated steady feature, unstated because it is impossible to make precise in the Euclidean picture. Translating this into Lorentzian space-time we see that F should be a map from $S^2 \times W \rightarrow T$, where W is the world line of the Source. Then the parity of the number of images of F at $y \epsilon T$ is equal to the local degree of F at y. In general this degree need not be odd. In the regimes considered in Ref. 7, however, the degree can be shown to be one. More details will appear in my article in the Proceedings of the 7th Marcel Grossman Conference.

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[6] P. Schneider, J. Ehlers, and E. E. Falco, *Gravitational Lenses* (Springer-Verlag, Berlin, 1992).
[7] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity* (Academic, New York, 1983), see Corollary 57, p. 89 or pp. 207–209.

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