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Proceedings of the American Mathematical Society
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HOMOLOGY TANGENT BUNDLES AND UNIVERSAL BUNDLES

DANIEL HENRY GOTTLIEB

ABSTRACT. We find results about the evaluation map from the group of homeomorphisms of a closed manifold $M$ and also about fibre bundles where $M$ is the fibre. These facts follow from the observation that the homology tangent bundle is induced from a universal bundle pair.

1. Introduction. The object of this paper is to prove the following theorem:

**Theorem 12.** Let $G$ be any group of homeomorphisms acting on a closed oriented topological manifold $M$ and let $\omega: G \to M$ be the evaluation map at the base point $\ast$. Then $\chi(M)\omega^*: \tilde{H}^*(M; R) \to \tilde{H}^*(G; R)$ is trivial where $R$ is any ring of coefficients with a unit and $\chi(M)$ is the Euler-Poincaré number of $M$.

Note that the theorem applies to the coset mapping $\rho: G \to G/H$.

The proof of Theorem 12 makes use of the following observation: Let $G$ be the path-connected component of the group of homeomorphisms of any topological manifold $M$ and let $H$ be the isotropy subgroup at $\ast$. Then we have the fibre bundle $M \to B_H \to B_G$.

**Theorem 5.** The inclusion $i$ induces the homology tangent bundle from a "universal" fibre pair over $B_H$.

We use a theorem of R. F. Brown to obtain Theorem 12 from Theorem 5. Along the way we obtain a topological version of a theorem of Borel [2].

**Theorem 9.** Let $M \to E \to B$ be any fibre bundle with an orientation preserving structural group, where $M$ is a closed, orientable, topological manifold. Let $p$ be a prime such that $\chi(M) \not\equiv 0 \mod p$. Then $\pi^*: H^*(B; \mathbb{Z}_p) \to H^*(E; \mathbb{Z}_p)$ is injective.
2. **Preliminaries.** Let $G$ be a topological group and let $H$ be a closed subgroup. Let $E$ be a space on which $G$ operates on the right such that the projections $E \rightarrow E/G$ and $E \rightarrow E/H$ are principal fibre bundles with fibres $G$ and $H$ respectively. Also, we have the fibre bundle $\rho : E/H \rightarrow E/G$ with fibre $G/H$.

Let $G$ operate on the left of a space $F$. Then, as usual, $E \times_G F$ will denote the quotient space of $E \times F$ under the equivalence relation $(e, x) \sim (eg, g^{-1}x)$. The equivalence class of $(e, x)$ will be denoted by $\langle e, x \rangle$. Thus $\langle e, x \rangle$ is a point of $E \times_G F$ and $\langle e, x \rangle = \langle eg, g^{-1}x \rangle$. Similarly, we shall denote points of $E/H$ by $\langle e \rangle_H$ where $e \in E$ and points of $E/G$ will be denoted $\langle e \rangle_G$.

The map $p : E \times_G F \rightarrow E/G$ given by $p(\langle e, x \rangle) = \langle e \rangle_G$ is the projection of a fibre bundle with fibre $F$ and group $G$.

Also, we need the concept of bundles induced by a map. Let $E \rightarrow^p B$ be a bundle and $X \rightarrow^f B$ be a map. The induced bundle $f^*(X) \rightarrow^p f(X)$ is a bundle with the same fibre and $f^*(X)$ is the subspace of $E \times X$ given by points of the form $(e, x)$ such that $p(e) = f(x)$. We shall denote points in $f^*(X)$ by $(e, x)$. Then the projection map $p(f)$ is given by $(e, x) \mapsto x$.

We wish to consider the square of the bundle $E/H \rightarrow^p E/G$. This is the bundle $G/H \rightarrow^p \rho^*(E/H) \rightarrow E/H$. The points of $\rho^*(E/H)$ have the form $\langle \langle e \rangle_H, \langle e' \rangle_H \rangle$ where $\langle e \rangle_G = \langle e' \rangle_G$. Note there is a unique $g \in G$ such that $e' \cdot g = e$.

There is another bundle with fibre $G/H$ over $E/H$. This is $G/H \rightarrow E \times_H G/H \rightarrow E/H$. These two bundles are equivalent. We record some standard facts below for the reader’s convenience:

**LEMMA 1.** *The two bundles $\rho^*(E/H) \rightarrow E/H$ and $E \times_H G/H \rightarrow E/H$ are equivalent. The bundle equivalence is given by the map $p^*(E/H) \rightarrow E \times_H G/H$ which sends $\langle \langle e \rangle_H, \langle e' \rangle_H \rangle \mapsto \langle e, g^{-1}H \rangle$ where $g$ is defined by the equation $e' \cdot g = e$.*

**LEMMA 2.** *There exists a cross-section $s : E/H \rightarrow \rho^*(E/H)$ to the fibration $\rho^*(E/H) \rightarrow E/H$ given by $\langle e \rangle_H \mapsto \langle \langle e \rangle_H, \langle e \rangle_H \rangle$. The fibration restricts over the fibre $G/H$ of $E/H \rightarrow^p E/G$ to projection on the second factor $G/H \times G/H \rightarrow G/H$. Then the cross-section $s$ restricted to the fibre $G/H$ is the diagonal $G/H \rightarrow G/H \times G/H$. We may also regard $s$ as the cross-section $E/H \rightarrow E \times_H G/H$ given by $\langle e \rangle_H \mapsto \langle e, H \rangle$.***

**LEMMA 3.** *There is a commutative diagram of fibre bundles

\[
\begin{array}{ccc}
G/H \times G/H & \rightarrow & \rho^*(E/H) \rightarrow E/H \\
\Delta \downarrow & & \downarrow p \\
G/H & \rightarrow & E/H & \rightarrow E/G \\
\end{array}
\]*
Lemma 4. There is a bundle equivalence

\[
\begin{array}{c}
E/H \xrightarrow{\mu} E \times_G G/H \\
\downarrow \rho \downarrow \downarrow \rho \\
E/G \xrightarrow{1} E/G
\end{array}
\]

where \(\mu((e)_H) = (e, H)\).

3. The homology tangent bundle. In this section we shall always assume that \(G/H\) is a topological manifold without boundary, which we shall also denote by \(M\). Also, \(G\) will be a group of homeomorphisms of \(M\) onto itself, endowed with the compact open topology. Then \(H\) will be a space of homeomorphisms of \(M\) which leave the base point \(* \in M\) fixed, i.e. the isotropy subgroup. We shall always take \(E\) to be contractible. Thus we have \(B_G = E/G\) and \(B_H = E/H\).

Now observe that \(H\) may be regarded as a group of homeomorphisms of \(M -> *\). Consider the universal principal bundle \(H \to E \to B_H\). By regarding \(H\) as operating on \(M\), we obtain the associated bundle \(M \to E \times_H M \to B_H\).

By regarding \(H\) as operating on \(M -*\) we obtain the associated bundle \((M -*) \to E \times_H (M -*) \to B_H\).

Observe that \(E \times_H (M -*)\) is contained as a subspace in \(E \times_H M\). In fact, using the cross-section \(s\) of Lemma 2, we have \(E \times_H (M -*) = (E \times_H M) - s(B_H)\). The pair \((E \times_H M, E \times_H (M -*))\) is a fibre-bundle pair in the sense of Fadell [6]. See also [10, p. 256]. The fibre is \((M, M -*)\) and the base is \(B_H\).

By Lemma 1, we may view the fibre pair from a different point. Recall the fibre bundle \(M \to B_H \to \rho B_G\). Then \(E \times_H M\) is just \(\rho^*(B_H)\), and \(E \times_H (M -*)\) is \(\rho^*(B_H) - s(B_H)\). Thus it is easy to see, using Lemma 3, that the fibre bundle \(M \times M \to \partial M\) given by projection on the second factor is induced by \(i: M \to B_H\) from the bundle \(\rho^*(B_H)\). Also, the bundle \(M \times M \to \partial M\) given by projection on the second factor is induced by \(i: M \to B_H\) from the bundle \(\rho^*(B_H) - s(B_H)\). We formalize this in the following theorem.

Theorem 5. The bundle pair \((M \times M, M \times M \to \partial M\) with fibre \((M, M -*)\) is induced from the bundle pair \((\rho^*(B_H), \rho^*(B_H) - s(B_H))\) by the inclusion map \(i: M \to B_H\).

Now let \((E', E'_0)\) be a fibre pair with fibre \((F, F_0)\) and base space \(B\). Then \(\pi_1(B, *)\) operates on \(H_\bullet(F, F_0; Z)\). The fibre pair \((E', E'_0)\) is said to be orientable if \(\pi_1(B, *)\) operates trivially on \(H_\bullet(F, F_0; Z)\). If \(\pi_1(B, *)\) operates trivially on \(H_\bullet(F, F_0; Z_2)\), the fibre pair is called \(Z_2\)-orientable.
Lemma 6. (a) The homology tangent bundle

\[(M, M - \ast) \to (M \times M, M \times M - \Delta) \xrightarrow{p_2} M\]

is \(\mathbb{Z}_2\)-orientable. If \(M\) is orientable, then the homology tangent bundle is orientable.

(b) Let \(M\) be orientable and let \(G\) be a group of homeomorphisms of \(M\) which preserve orientation. Then the fibre pair

\[(M, M - \ast) \to (\rho^*(B_H), \rho^*(B_H) - s(B_H)) \to B_H\]

is orientable.

(c) For any \(M\) and \(G\), the above fibre pair is \(\mathbb{Z}_2\)-orientable.

Proof. (a) was proved by Fadell in [6].

(b) Let \(\alpha \in \pi_1(B_H, \ast)\). We shall show that \(\alpha\) acts trivially on \(H_n(M, M - \ast)\). There exists an isotopy \(f_t: M \to B_H\) such that:

1. each \(f_t\) is a homeomorphism of \(M\) onto a fibre of \(B_H \to \rho B_H\).
2. \(f_0\) is the identity map of \(M\) regarded as the fibre over \(p(\ast)\).
3. the trace of the isotopy (i.e. the path \(\sigma: t \to f_t(\ast)\)) is a closed path representing \(\alpha\).

Now we define an isotopy \(g_t: M \to \rho^*B_H\) by setting \(g_t(m) = (f_t(m), f_t(\ast)) \in \rho^*B_H\). Note that \(g_t\) is an isotopy of the pairs \((M, M - \ast) \to (\rho^*B_H, \rho^*B_H - s(B_H))\).

Now \(g_0\) is the identity on the fibre \((M, M - \ast)\) over \(\ast \in B_H\). On the other hand \(g_1(m) = (f_1(m), \ast) \in (M, M - \ast)\). Since \(f_1\) preserves orientation on \(M\) by hypotheses, we see that \(g_1: (M, M - \ast) \to (M, M - \ast)\) must induce the identity homomorphism on \(H_n(M, M - \ast)\). Thus \(\alpha\) acts trivially on \(H_n(M, M - \ast)\), which was to be shown.

(c) is obvious since \(H_n(M, M - \ast; \mathbb{Z}_2) \cong \mathbb{Z}_2\).

Let \((E', E_0') \to B\) be an orientable fibre bundle pair with fibre \((M, M - \ast)\). Since \(H^*(M, M - \ast) = H^*(R^n, R^n - 0)\), there exists a "Thom isomorphism" \(\phi: H^*(B) \cong H^*(E', E_0')\) which is natural with respect to mappings of fibre bundle pairs. Then \(\phi(1) \in H^*(E', E_0'; R)\), where \(R\) is a ring of coefficients, is the "Thom class". Characteristic classes (Fadell [6]) are defined from the Thom class in the obvious way. For example, \(\phi^{-1}(\phi(1) \cup \phi(1))\) is the Euler class.

The characteristic classes of \(M\) are defined to be the characteristic classes of the homology tangent bundle \((M, M \times M - \Delta)\).

Corollary 7. If \(k \in H^i(M; R)\) is a characteristic class of \(M\), then \(k\) is in the image of \(i^*: H^i(B_H; R) \to H^i(M; R)\).

Now we assume that \(M\) is a closed orientable \(n\)-manifold. Then R. F. Brown [4] says the Euler class of \(M\) is equal to \(\chi(M)\mu\) where \(\mu \in H^n(M; R)\).
is the fundamental class of the manifold and $\chi(M)$ is the Euler-Poincaré number of $M$. If $\varepsilon \in H^n(B_H; R)$ is the Euler class for the fibre pair $(\rho^*(B_H), \rho^*(B_H) - s(B_H)) \rightarrow B_H$, then by Theorem 5 we have $i^*(\varepsilon) = \chi(M) \mu$.

**Corollary 8.** $i^*(\varepsilon) = \chi(M) \mu$.


**Theorem 9.** Let $M \rightarrow E \rightarrow^p B$ be a fibre bundle with orientation preserving structural group and let $M$ be a closed, orientable topological manifold. If $\chi(M) \not\equiv 0 \pmod{p}$, then $p^*$ is injective. In addition, the theorem is still true when $p=2$ with no orientability condition on $M$ or on the fibre bundle.

**Proof.** We know that the Euler class of $M$ is $\chi(M) \mu \in H^n(M; Z)$, where $\mu$ is a generator, by a theorem of R. F. Brown [4]. Let $\chi(M) \mu$ also denote the element in $Z_\mu$ cohomology. If $\chi(M) \not\equiv 0 \pmod{p}$, then

$$\chi(M) \mu \not\equiv 0 \in H^n(M; Z_\mu).$$

Consider the fibration $M \rightarrow^i B_H \rightarrow^p B_G$. By Corollary 8, $\chi(M) \mu$ is in the image of $i^*: H^n(B_H; Z_\mu) \rightarrow H^n(M; Z_\mu)$. (Note that in order to apply Corollary 8 we need that Lemma 6 be valid. But Lemma 6(b) requires that $G$ be orientation preserving.) For $Z_\mu$ coefficients we apply (c). By naturality, $\chi(M) \mu$ is in the image of $i^*$ for the fibration $M \rightarrow^i E \rightarrow^p B$.

Lemma 3.1 of [2] (or see Theorem 14.5 of [3]) says the following: Suppose $F \rightarrow^i E \rightarrow^p B$ is a fibration with an $n$-dimensional fibre $F$ such that $\pi_1(B)$ acts trivially on $H^n(F; Z_\mu)$. If $i^*: H^n(E; Z_\mu) \rightarrow H^n(F; Z_\mu)$ is nonzero, then $p^*$ is injective.

In the case at hand, $\pi_1(B)$ operates trivially on $H^n(M; Z_\mu)$ since $\pi_1(B)$ operates trivially on the image of $i^*$ (which is onto $H^n(M; Z_\mu)$).

**Remark.** In a forthcoming paper, we shall remove the hypothesis that $M$ is orientable.

### 5. Actions and characteristic classes.

In this section we shall prove that a characteristic class of a topological manifold $M$ is "inert" under the action $\phi: G \times M \rightarrow M$. We say $k \in H^*(M; R)$ is inert under $\phi$ if $\phi^*(k) = 1 \times k$, where $R$ is any ring of coefficients. One consequence of the inertness of characteristic classes is that $\chi(M) \omega^*$ is trivial where $\omega: G \rightarrow M$ is evaluation at a base point $\ast$.

We begin by studying the action $\phi: G \times (G/H) \rightarrow G/H$ given by $\phi(g, g' H) = gg'H$. This action fits inside the following commutative diagram.
\[ G \times (G/H) \xrightarrow{\dot{\phi}} G/H \]
\[ \downarrow i \times 1 \downarrow i \]
\[ (*) \]
\[ E \times (G/H) \xrightarrow{\phi} B_H = E/H \]
\[ \downarrow p \times * \downarrow p \]
\[ B_G \xrightarrow{1} B_G \]

where \( \phi \) is defined by \( \phi(e, gH) = (eg)_H \) and \( i: G \to E \) is given by \( i(g) = g \) and \( i: G/H \to E/H \) is given by \( i(gH) = \langle g \rangle_H \). Observe that \( \phi \) is well defined and continuous and that (*) in fact commutes.

Since \( E \) is contractible, the commutativity of (*) tells us that \( i \circ \dot{\phi} \) is homotopic to \( G \times (G/H) \to^{\text{proj}} G/H \to^i B_H \). This proves the following theorem.

**Theorem 10.** Any \( k \) which is in the image of \( i^*: H^*(B_H; R) \to H^*(G/H; R) \) is inert under the action \( \dot{\phi} \). In particular, characteristic classes of topological manifolds are inert under \( \dot{\phi} \).

**Remark.** The inertness of characteristic classes of differentiable manifolds under differentiable actions will be shown in [9] (i.e. Stiefel-Whitney classes and Pontrjagin class are inert under differentiable actions).

The usefulness of the concept of inertness comes from the following lemma. Let \( \omega: G \to M \) be given by \( \omega = \dot{\phi}(\cdot, *) \) for base point \( * \in M \).

**Lemma 11.** Suppose that \( u \) and \( v \) are positive dimensional cohomology elements of a space \( M \), and suppose that \( v \) is inert under some action \( \dot{\phi} \). Then \( u \cup v = 0 \) implies \( \omega^*(u) \times v = 0 \).

**Proof.** We have
\[
0 = \dot{\phi}^*(u \cup v) = \dot{\phi}^*(u) \cup \dot{\phi}^*(v)
\]
\[
= (\omega^*(u) \times 1 + \sum a_i \times b_i + \sum c_i \times d_i) \cup 1 \times v
\]
\[
= \omega^*(u) \times k + \sum a_i \times (b_i \cup k) + \sum (c_i \times d_i) \cup (1 \times v).
\]

Here the \( a_i \) and \( c_i \in H^*(M; R) \) are positive dimensional and \( b_i \) and \( d_i \in H^*(G; R) \) and \( \times \) is the cohomology cross product and \( * \) is the torsion product coming from the Künneth formula. It is easy to see that no term in the expansion has the right dimensions to cancel \( \omega^*(u) \times k \). Hence \( \omega^*(u) \times k = 0 \).

The above lemma, with the aid of Corollary 8 gives us the main result.
Theorem 12. Let $M$ be a closed orientable topological manifold and let $G$ be a group of homeomorphisms acting on $M$ by the action $\cdot: G \times M \to M$. Let $\omega: G \to M$ be the evaluation map at the base point $\ast$. Then $\chi(M)\omega^* : \tilde{H}^*(M; R) \to \tilde{H}^*(G; R)$ is trivial where $R$ is any coefficient ring with unit.

Proof. By Corollary 8, $i^*(e) = \chi(M)\mu$. So $\chi(M)\mu$ is inert by Theorem 10. (We let $\chi(M)\mu$ also stand for the image of $\chi(M)\mu$ in cohomology with coefficients in $R$.) Since $\mu$ is the top dimensional class, $\mu \cup \nu = 0$ for any $\nu \in \tilde{H}^*(M; R)$. Thus, by Lemma 11,

$$0 = \omega^*(\nu) \times \chi(M)\mu = \chi(M)\omega^*(\nu) \times \mu.$$

So $\chi(M)\omega^*(\nu)$ must equal zero, thus proving the theorem.

Remark. If $R = \mathbb{Z}_2$, the orientability requirement may be dropped.

Bibliography


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