The Sullivan Conjecture

The Sullivan Conjecture in Homotopy Theory usually refers to a theorem about the contractibility or homotopy equivalence of certain types of mapping spaces. These results are vast generalizations of two different but related conjectures made by Dennis Sullivan in 1972.

Haynes Miller achieved the first major breakthrough and is given credit for solving the Sullivan conjecture. This was published in 1984 and one version reads: The space of pointed maps $Map_*(B_G, X)$ from the classifying space of of a finite group G to a finite CW-complex X is weakly contractible. The mapping space has the compact-open toplogy.

An equivalent statement: The space of unpointed maps $Map(B_G, X)$is weakly homotopy equivalent to X (under the same hypotheses on G and X.

These theorems are still true when B_G is replaced by a CW-complex which has only finitely many non-zero homotopy groups, each of which is locally finite and where X can be any finite dimensional CW-complex. This improvement is due to Alexander Zabrodsky.

Equivariant versions of the Sullivan Conjecture come about by considering the question: How close does the natural map $X^G \to X^{hG}$ come to being a homotopy equivalence? Here X^G is the fixed point set of a group action G on the space X and the homotopy fixed point set is $X^{hG} = Map_G(E_G, X)$, the space of equivariant maps from the contractible space E_G on which G acts freely to the G-space X. For G acting trivially on X, we see that Miller's version of the Sullivan Conjecture gives a positive answer to this question.

Another version of this question is that the fixed point set of a G-space localized at a prime p is weakly homotopy equivalent to the homotopy fixed point set of G acting on the p-localization of X. One proof of this result is via the Segal Conjecture by Gunnar Carlsson. Haynes Miller also independently proved this result, and J. Lannes has a subsequent proof using his T functor.

These theorems have found many beautiful applications at the hands of the above mentioned workers as well as W. G. Dwyer, C. McGibbon, J. A. Neisendorfer and C. Wilkerson, S. Jackowsky to mention only a few.

References:

Miller, Haynes The Sullivan conjecture and homotopical representation theory. Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), 580–589, Amer. Math. Soc., Providence, RI, 1987.

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