1. The stock for Mao Manufacturing LTD pays quarterly dividends. The next dividend will be 2.10 and will be paid in two months. Each dividend will be 0.30 greater than the prior dividend. In other words, the first dividend in two months will be 2.10. The second dividend at the end of five months will be 2.40. The third dividend at the end of 8 months will be 2.70. This pattern will continue forever.

Find the price of Mao stock using the dividend discount method and an interest rate of 9% compounded quarterly.

Solution:

Since the first dividend is paid in two months, the dividend discount method would put the present value at time -1 month.

We will have to accumulate the present value by 1 month to set it at time 0.

\[ \frac{i^{(4)}}{4} = \frac{0.09}{4} = 0.0225 \]

\[ PV = \left( \frac{P}{i} + \frac{Q}{i^2} \right)(1 + \frac{i^{(4)}}{4}) = \left( \frac{2.10}{0.0225} + \frac{0.30}{(0.0225)^2} \right)(1.0225)^{1/3} = 691.0322622 \]
2. Calculate the Macaulay Convexity for a four year annuity due with level annual payments of 1000 using an interest rate of 4.5%.

Solution:

\[ i = 0.045 \]

\[
C(i, \infty) = \sum C_i \frac{v^t}{t^2} = \frac{1000v^0 + 1000v^1 + 1000v^2 + 1000v^3}{1000v^0 + 1000v^1 + 1000v^2 + 1000v^3}
\]

\[
= \frac{1000v + 4000v^2 + 9000v^3}{1000 + 1000v + 1000v^2 + 1000v^3} = \frac{12506.52704}{3748.964354} = 3.335995187
\]
3. You are given the following spot interest rate yield curve:

<table>
<thead>
<tr>
<th>Time (t)</th>
<th>Spot Rate ($r_t$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>2.7%</td>
</tr>
<tr>
<td>1.0</td>
<td>3.1%</td>
</tr>
<tr>
<td>1.5</td>
<td>3.5%</td>
</tr>
<tr>
<td>2.0</td>
<td>4.0%</td>
</tr>
<tr>
<td>2.5</td>
<td>4.5%</td>
</tr>
<tr>
<td>3.0</td>
<td>5.0%</td>
</tr>
<tr>
<td>3.5</td>
<td>5.6%</td>
</tr>
<tr>
<td>4.0</td>
<td>6.1%</td>
</tr>
</tbody>
</table>

Sandra is the beneficiary of a 3 year annuity due with annual payments. The first payment is 1000. The second payment is 2000. The third payment is 4000.

Calculate the accumulated value of this annuity at the end of 3 years.

Solution:

To find the accumulated value, first find the present value and then accumulate it forward 3 years.

\[
PV = 1000 + 2000\left(\frac{1}{1 + r_1}\right) + 4000\left(\frac{1}{1 + r_2}\right)^2 = 1000 + 2000\left(\frac{1}{1.031}\right) + 4000\left(\frac{1}{1.04}\right)^2
\]

\[
= 6638.089062
\]

\[
AV = PV(1 + r_3)^3 = 6638.089062(1.05)^3 = 7684.41785
\]
4. Wang Bank has a portfolio of bonds which have the following characteristics at an interest rate of 7%:

   i. The price of the portfolio is 890,000.
   ii. The Macaulay Duration of the portfolio is 6.
   iii. The Macaulay Convexity of the portfolio is 32.

Using both the Duration and the Convexity of the portfolio, Wang wants to estimate the price of the portfolio of bonds if the interest rate changed to 6%.

Determine the estimated price at 6%.

**Solution:**

\[
P(i) = P(i_0)[1 - D(i,1)(i - i_0) + C(i,1)(\frac{(i - i_0)^2}{2})]
\]

\[
P(0.06) = P(0.07)[1 - (\frac{6}{1.07})(0.06 - 0.07) + (\frac{6+32}{1.07^2})(\frac{(0.06 - 0.07)^2}{2})]
\]

\[
= 890000[1 + 0.056074766 + 0.001659534]
\]

\[
= 941,383.5266
\]
5. A 30 year bond has annual coupons of 400 and a maturity value of 7000. Calculate the Modified Duration of this bond using an annual interest rate of 8%.

Solution:

\[
D(i, \infty) = \frac{\sum C_i \nu^i}{\sum C_i \nu^i} = \frac{400a_{30} + \frac{400}{0.08} (a_{30} - 30\nu^{30}) + 7000(30)\nu^{30}}{N=30, \ I/Y=8, \ PMT=400, \ FV=7000, \ CPT \ PV}
\]

\[
= \frac{400(11.25778334) + 5000(11.25778334 - 2.981319976) + 20869.23894}{5198.754665} = 12.84051151
\]

\[
D(i, 1) = \frac{D(i, \infty)}{1+i} = \frac{12.84051151}{1.08} = 11.88936251
\]
6. The Kinney Investment Bank has a portfolio of two liabilities:
   
   i. The first one is a payment of 256,000 to Grant at the end of 5 years.
   
   ii. The second one is bond that was sold to Matt. At an interest rate of 6%, Matt’s bond has a price of 72,000 with a Macaulay Duration of 16.

Kinney Investment Bank also owns a zero coupon bond which matures for 100,000 at the end of three years.

Kinney wants to spend 180,000 to purchase another bond. After this purchase, Kinney wants the duration of the two assets in its portfolio to equal the duration of the portfolio of liabilities if all calculations are done using an annual interest rate of 6%.

If Kinney is to accomplish this objective, determine the Macaulay Duration of the bond that Kinney must purchase.

**Solution:**

Duration of Portfolio Assets = Duration of Portfolio Liabilities

Let Liability1 = A, Let Liability2 = B, Let ZCB = C, Let new bond = D

\[
\frac{D_CPV_C + D_DPV_D}{PV_C + PV_D} = \frac{D_APV_A + D_BPV_B}{PV_A + PV_B}
\]

\[
\frac{3(100,000v^3) + D_D(180,000)}{100,000v^3 + 180,000} = \frac{5(256,000v^5) + 16(72,000)}{256,000v^5 + 72,000}
\]

\[
251885.7849 + 180,000D_D \approx 263961.9283
\]

\[
D_D = 10.34400374
\]
7. Yacko Insurance Company has promised to pay Jake $100,000 at the end of one year. Yacko has also agreed to pay $200,000 at the end of two years and $150,000 at the end of three years to Jake.

Alexandra, Yacko’s Chief Actuary, wants to exactly match the liability cash flows using the following three bonds:

i. A one year bond with an annual coupon of 50 and a maturity value of 1000.

ii. A two year bond with annual coupons of 60 and a maturity value of 1000.

iii. A three year bond with annual coupons of 70 and a maturity value of 1000.

Calculate the number of one year bonds that Alexandra should purchase assuming partial bonds can be purchased.

**Solution:**

Let $a = \#$ of one year bonds, Let $b = \#$ of two year bonds, Let $c = \#$ of three year bonds

\[
100000 = 1050a + 60b + 70c \\
200000 = 1060b + 70c \\
150000 = 1070c \\
c = 140.1869159 \\
b = 179.4216188 \\
a = 75.63963691
\]
8. The following three bonds may each be purchased for 1050:

   i. A one year bond with an annual coupon of 100 and a maturity value of 1000.
   ii. A two year bond with annual coupons of 150 and a maturity value of 925.
   iii. A three year bond with annual coupons of 200 and a maturity value of 800.

Calculate the three year spot interest rate.

Solution:

\[ 1050 = 1100(1 + r_1)^{-1} \]
\[ r_1 = 0.047619048 \]
\[ 1050 = 150(1 + r_1)^{-1} + 1075(1 + r_2)^{-2} \]
\[ r_2 = 0.08879 \]
\[ 1050 = 200(1 + r_1)^{-1} + 200(1 + r_2)^{-2} + 1000(1 + r_3)^{-3} \]
\[ r_3 = 0.131454648 \]
9. The Rotter Company has promised to pay its president, Thomas, a bonus of 1,000,000 at the end of 10 years. Rotter wants to protect itself from interest rate changes using Reddington Immunization. It will do so by buying the following two bonds:

i. Bond A is a zero coupon bond which matures for 10,000 at the end of 4 years.

ii. Bond B is a zero coupon bond which matures for 15,000 at the end of 15 years.

The current interest rate is 8.5%.

Calculate the number of each bond that Rotter will buy assuming partial bonds can be purchased.

**Solution:**

**Present Value Matching:**
Let $A =$ Present value of 4 year bond
Let $B =$ Present value of 15 year bond
$\text{PV(Assets)} = \text{PV(Liabilities)}$
$A + B = 1,000,000v^{10}$

**Duration Matching:**
$\text{Duration(Assets)} = \text{Duration(Liabilities)}$
$4A + 15B = 10(1,000,000v^{10})$

Solve for $A$ & $B$:
$A = 1,000,000v^{10} - B$
$4(1,000,000v^{10} - B) + 15B = 10,000,000v^{10}$
$4,000,000v^{10} - 4B + 15B = 10,000,000v^{10}$
$11B = 6,000,000v^{10}$
$B = 241,246.59$

$A = 201,038.825$

Remember $A$ & $B$ represent the total present value of the zero coupon bonds.
$201,038.825 = (# A)(10,000v^{4})$
$# A = 27.86114048$
$241,246.59 = (# B)(15,000v^{15})$
$# B = 54.67842509$
10. You are given the following forward interest rates:

   i. $f_{[0,1]} = 0.05$

   ii. $f_{[1,2]} = 0.06$

   iii. $f_{[2,3]} = 0.07$

Calculate the price of a 3 year bond with annual coupons of 150 and a maturity value of 1000.

**Solution:**

\[(1 + r_1) = (1 + f_{[0,1]}) = (1.05)\]

\[r_1 = 0.05\]

\[(1 + r_2)^2 = (1 + f_{[0,1]})(1 + f_{[1,2]}) = (1.05)(1.06)\]

\[r_2 = 0.054988152\]

\[(1 + r_3)^3 = (1 + f_{[0,1]})(1 + f_{[1,2]})(1 + f_{[2,3]}) = (1.05)(1.06)(1.07)\]

\[r_3 = 0.059968553\]

Price = \[150(1 + r_1)^{-1} + 150(1 + r_2)^{-2} + 1150(1 + r_3)^{-3}\]

= 1243.276149

*OR*

Price = \[
\frac{150}{(1.05)} + \frac{150}{(1.05)(1.06)} + \frac{1150}{(1.05)(1.06)(1.07)} = 1243.27615
\]
11. Taylor is receiving an annuity immediate with non-level payments. The first payment is 10,000 at the end of one year. The second payment is 40,000 at the end of 3 years. The final payment is 25,000 at the end of 5 years.

Calculate the Macaulay Duration of Taylor’s annuity using an annual interest rate of 10%.

Solution:

\[
D(i, \infty) = \frac{10000(1)v^1 + 40000(3)v^3 + 25000(5)v^5}{10000v^1 + 40000v^3 + 25000v^5} = \frac{176863.8506}{54666.5342} = 3.23532218
\]
12. The stock of Shi Corporation pays quarterly dividends with the next dividend payable later today. The dividend today will be 10. Each future dividend will be 2% higher than the prior dividend. In other words, the first dividend will be 10. The second dividend will be \(10(1.02)\). The third dividend will be \(10(1.02)^2\), etc.

Using the dividend discount method, calculate the price of Shi stock to yield a rate of 20% compounded quarterly.

Solution:

\[ PV = 10 + 10(1.02)v + 10(1.02)^2 v^2 + ... \]

\[ = \frac{10 - 0}{1 - 1.02v} = 350 \]

\[ v = \frac{1}{1 + \frac{0.20}{4}} = \frac{1}{1.05} \]