1. (6 points) Yifei who is (45) is receiving an annuity with payments of 25,000 at the beginning of each year. The annuity guarantees that payments will be made for 15 years. Thereafter payments will only be made if Yifei is alive.

You are given that mortality follows the Illustrative Life Table and interest is at 6%.

Calculate the Actuarial Present Value of Yifei’s annuity.

\[ APV = 25,000 \cdot \ddot{a}_{\overline{45}|15} \]
\[ = 25,000 \cdot \left( \ddot{a}_{15} + E_{45} \cdot \ddot{a}_{60} \right) \]
\[ = 25,000 \cdot \left( \frac{1 - v^{15}}{1 - v} + v^{15} \cdot \frac{l_{60}}{l_{45}} \cdot \ddot{a}_{60} \right) \]
\[ = 25,000 \times \left( \frac{1 - 1.06^{-15}}{1 - 1.06^{-1}} + 1.06^{-15} \times \frac{8,188,074}{9,164,051} \times 11.454 \right) \]
\[ = 361,256.9925 \]
2. (12 points) Spears Life Insurance Company has 500 whole life annuities issued to (62). Each annuitant is independent. The annuity pays an annual benefit of 1000 at the beginning of each year for as long as the annuitant is alive.

You are given that mortality follows the Illustrative Life Table and interest is at 6%.

Spears set aside an amount of $5,250,000 to pay for the future benefits under these annuities.

Using the Normal Distribution, calculate the probability that the $5,250,000 will be sufficient to cover all the benefits that will be paid by Spears.

Say, the Random Variable of PV of Benefit of the $i^{th}$ annuitant is denoted as $Y_i$.

$E[Y_i] = 1000 \cdot \bar{a}_{62} = 1000 \times 10.6584 = 10,658.4$

$Var(Y_i) = \left( \frac{1000}{d} \right)^2 \cdot \left[ A^2_{62} - \left( A_{62} \right)^2 \right]$

$= \frac{1000^3}{\left(1 - 1.06^{-1} \right)^2} \times \left( 0.19941 - 0.39670^2 \right) = 13,120,873.33$

Since each annuitant is independent, PV of Benefit paid to all annuitants, say $Y$, is approximately normal distributed with

$E(\text{Port}) = 500 \cdot E[Y_i] = 500 \times 10.658.4 = 5,329,200$

$Var(\text{Port}) = 500 \cdot Var(Y_i) = 500 \times 13,120,873.33 = 6,560,436,666$

$Pr[\text{TotalPayments } < 5,250,000]$

$= Pr \left( Z < \frac{5,250,000 - E(\text{Port})}{\sqrt{Var(\text{Port})}} = \frac{5,250,000 - 5,329,200}{\sqrt{6,560,436,666}} \approx -0.98 \right)$

$= 1 - Pr( Z < 0.98 )$

$= 1 - 0.8365$

$= 0.1635$
3. Peterson Pet Insurance Company has developed the following life insurance table for dogs:

<table>
<thead>
<tr>
<th>Age</th>
<th>( l_x )</th>
<th>Age</th>
<th>( l_x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2000</td>
<td>5</td>
<td>1200</td>
</tr>
<tr>
<td>1</td>
<td>1950</td>
<td>6</td>
<td>1000</td>
</tr>
<tr>
<td>2</td>
<td>1850</td>
<td>7</td>
<td>700</td>
</tr>
<tr>
<td>3</td>
<td>1600</td>
<td>8</td>
<td>300</td>
</tr>
<tr>
<td>4</td>
<td>1400</td>
<td>9</td>
<td>0</td>
</tr>
</tbody>
</table>

You are given that \( v = 0.9 \).

Rachit wants to buy a whole life policy which pays a death benefit of 2000 at the end of the year of death for his dog who is age 6. Rachit will pay an annual premium for this coverage.

Rachit, an actuarial student at Purdue University, estimates that the net annual benefit premium for this policy should be 870. Rachit is accurate to within the nearest 10 in his calculation of the net benefit premium.

a. (4 points) Calculate the exact net benefit premium.

\[
d_6 = l_6 - l_7 = 1000 - 700 = 300 \\
d_7 = l_7 - l_8 = 700 - 300 = 400 \\
d_8 = l_8 - l_9 = 300 - 0 = 300
\]

\[
PVP = PVB \\
P \cdot \hat{a}_6 = 2000 \cdot A_6
\]

\[
P \cdot \frac{1}{l_6} \left( l_6 + l_7 \cdot v + l_8 \cdot v^2 \right) = 2000 \cdot \frac{1}{l_6} \left( d_6 \cdot v + d_7 \cdot v^2 + d_8 \cdot v^3 \right)
\]

\[
\Rightarrow P = \frac{2000 \left( d_6 \cdot v + d_7 \cdot v^2 + d_8 \cdot v^3 \right)}{l_6 + l_7 \cdot v + l_8 \cdot v^2}
\]

\[
= \frac{2000 \times (300 \times 0.9 + 400 \times 0.9^2 + 300 \times 0.9^3)}{1000 + 700 \times 0.9 + 300 \times 0.9^2} \approx 867.80566
\]
b. (10 points) Calculate the $\text{Var}(L_0^n)$ where $L_0^n$ is the future loss random variable at time 0 for this policy using only benefits and net premiums. (If you cannot find the exact net benefit premium in part a., use Rachit’s estimated premium to calculate the variance.)

$$A_b = \frac{1}{l_6} \cdot \left(d_6 v + d_7 v^2 + d_8 v^3\right)$$

$$= \frac{1}{1000} \times \left(300 \times 0.9 + 400 \times 0.9^2 + 300 \times 0.9^3\right) = 0.8127$$

$$^2 A_b = \frac{1}{l_6} \cdot \left(d_6 v^2 + d_7 v^4 + d_8 v^6\right)$$

$$= \frac{1}{1000} \times \left(300 \times 0.9^2 + 400 \times 0.9^4 + 300 \times 0.9^6\right) \approx 0.66487$$

$$\text{Var}(L_0^n) = \left(S + \frac{P}{d}\right)^2 \cdot \left[^2 A_b - (A_b)^2\right]$$

$$= \left(2000 + \frac{867.80566}{1 - 0.9}\right)^2 \times \left(0.66487 - 0.8127^2\right)$$

$$\approx 500.666.8797$$

c. (5 points) Peterson decides to charge Rachit a gross premium payable annually of 1000. This gross premium will cover benefits and expenses of 10% of premium plus $5 per policy at the beginning of each year as well as provide a margin for profit.

Write an expression for $L_0^n$ which is the future loss random variable at time 0 for this policy considering benefits, expenses and premiums.

$$L_0^n = PVB + PVE - PVP$$

$$= 2000 \cdot v^{K_x+1} + 10\% \cdot G \cdot \ddot{a}_{K_x+1} + 5 \cdot \ddot{a}_{K_x+1} - G \cdot \ddot{a}_{K_x+1}$$

$$= 2000 \cdot v^{K_x+1} + (5 - 0.9 \cdot G) \ddot{a}_{K_x+1}$$

$$= 2000 \cdot v^{K_x+1} - 895 \cdot \ddot{a}_{K_x+1}$$
d. (12 points) Calculate the $\sqrt{\text{Var}(L_0^x)}$.

$L_0^x = 2000 \cdot v^{K,x+1} - 895 \cdot \bar{a}_{x+1}$

$= 2000 \cdot v^{K,x+1} - 895 \cdot \frac{1 - v^{K,x+1}}{d}$

$= \left(2000 + \frac{895}{d}\right) \cdot v^{K,x+1} - \frac{895}{d}$

$\text{Var}(L_0^x) = \left(2000 + \frac{895}{d}\right)^2 \cdot \text{Var}(v^{K,x+1})$

$= \left(2000 + \frac{895}{d}\right)^2 \left[2A_e - (A_e)^2\right]$  

$= \left(2000 + \frac{895}{1-0.9}\right)^2 \times (0.66487 - 0.8127^2)$

$\approx 526,493.0765$

$\sqrt{\text{Var}(L_0^x)} = \sqrt{526,217.30078} \approx 725.59843$
4. (7 points) Renee who is (20) purchases a special whole life insurance policy with a death benefit that changes over time. The death benefit which is payable at the end of the year of death is 50,000 if Renee dies before age 40. The death benefit is 120,000 if Renee dies after age 40.

You are given that mortality follows the Illustrative Life table with interest at 6%. You are also given that deaths are uniformly distributed between integral ages.

Calculate the Actuarial Present Value of Renee’s benefits.

\[
APV = 50,000 \cdot A_{20} + 70,000 \cdot E_{20} \cdot A_{40}
\]

\[
= 50,000 \cdot A_{20} + 70,000 \cdot 0.30193 \cdot 0.16132
\]

\[
= 6,673.51433
\]
5. (8 points) A whole life insurance policy to (70) has a death benefit of 100,000 payable at the moment of death. The policy has gross annual premiums payable for the life of the policy which are determined using the equivalence principle.

You are given that mortality follows the Illustrative Life Table with interest at 6%. Further, you are given that deaths are uniformly distributed between integral ages.

You are also given the following expenses:
   a. Commissions of 50% of premium in the first year and 8% of premium thereafter.
   b. Per Policy issue expense of 250.
   c. Maintenance expense of 30 per policy each year including the first year. The maintenance expense is incurred at the beginning of the year.
   d. A claim expense of 500 incurred at the moment of death.

Calculate the gross annual premium.

\[ PVP = PVB + PVE \]
\[ G \cdot \ddot{a}_{70} = 100,000 \cdot \overline{A}_{70} + 42\% \cdot G + 8\% \cdot G \cdot \ddot{a}_{70} + 250 + 30 \cdot \ddot{a}_{70} + 500 \cdot \overline{A}_{70} \]
\[ G \cdot (0.92 \cdot \ddot{a}_{70} - 0.42) = 100,500 \cdot \frac{i}{\delta} \cdot \overline{A}_{70} + 250 + 30 \cdot \ddot{a}_{70} \]
\[ G = \frac{100,500 \times 1.02971 \times 0.51495 + 250 + 30 \times 8.5693}{0.92 \times 8.5693 - 0.42} = 7,207.78118 \]
6. Dora who is the Chief Actuary for Zhang Life Insurance Company has been asked to calculate the net premium to be paid monthly for a whole life insurance on (50) with a death benefit of 75,000 payable at the moment of death.

Dora calls into her office two actuarial students – Xinyao and Shuang. She tells them that under the Equivalence Principle, you can calculate the net premium be setting the actuarial present value of premiums equal to the actuarial present value of benefits.

Dora also tells Xinyao and Shuang that she wants to assume that mortality follows the Illustrative Life Table and the interest is at 6%.

a. (4 points) Dora instructs Xinyao to calculate the present value of benefits. Xinyao calculates the actuarial present value of benefits assuming that deaths are uniformly distributed between integral ages.

Determine the Actuarial Present Value of the benefits as calculated by Xinyao.

\[
APVB = 75,000 \bar{A}_{50}
\]
\[
= 75,000 \cdot \frac{i}{\delta} \cdot A_{50}
\]
\[
= 75,000 \times 1.02971 \times 0.24905
\]
\[
= 19,233.69566
\]

b. (8 points) Dora tells Shuang that to calculate the actuarial present value of premiums she will need \( \bar{a}^{(12)}_{50} \). Dora asks Shuang to calculate \( \bar{a}^{(12)}_{50} \).

Shuang calculates \( \bar{a}^{(12)}_{50} \) using the three term Woolhouse formula.

Calculate the value of \( \bar{a}^{(12)}_{50} \).

\[
\mu_{50} \approx -\frac{1}{2} \left[ \ln(p_{50}) + \ln(p_{49}) \right]
\]
\[
\approx -\frac{1}{2} \times \left[ \ln(1 - 0.00592) + \ln(1 - 0.00546) \right] = 0.0057063
\]
\[
\bar{a}^{(12)}_{50} = \bar{a}_{50} - \frac{12 - 1}{2 \times 12} - \frac{12^2 - 1}{12 \times 12^2} \left( \delta + \mu_{50} \right)
\]
\[
= 13.2668 - \frac{11}{24} - \frac{143}{1728} \times \left[ \ln(1.06) + 0.0057063 \right]
\]
\[
\approx 12.80317
\]
c. (2 points) Using the work performed by Xinyao and Shuang, determine the net premium that Dora would calculate.

\[ \text{PVP} = \text{PVB} \]
\[ 12P \cdot \bar{a}_{50}^{(12)} = 75,000 \cdot \bar{A}_{50} \]
\[ P = \frac{75,000 \cdot \bar{A}_{50}}{12 \cdot \bar{a}_{50}^{(12)}} = \frac{19,233.69566}{12 \times 12.80317} \approx 125.18835 \]

d. (4 points) Explain the inconsistency in the approaches used by Xinyao and Shuang.

Xinyao is using an assumption of Uniform Death Distribution between integer ages.
Shuang is using the Woolhouse formula that is not based on an assumption of UDD.
7. (12 points) $Y$ is the present value random variable for a whole life annuity to (75) which pays 200 at the beginning of every year.

You are given that mortality follows the Illustrative Life table with interest at 7%. You are also given that deaths are uniformly distributed between integral ages.

Calculate $\Pr(Y > 1000)$.

$$Y = 200 \cdot \bar{a}_{x}$$

$$\Pr(Y > 1000)$$

$$= \Pr\left(\frac{200 \cdot (1 - v^{x+1})}{d} > 1000\right)$$

$$= \Pr\left(1 - v^{x+1} > 5d\right)$$

$$= \Pr\left(v^{x+1} < 1 - 5d\right)$$

$$= \Pr\left(K_x + 1 > \frac{\ln(1 - 5d)}{\ln(v)}\right)$$

$$= \Pr\left(K_x > \frac{\ln(1 - 5d)}{\ln(v)} - 1 = \frac{\ln(1 - 5 \times (1 - 1.07^{-1}))}{\ln(1.07^{-1})} - 1 \approx 4.855\right)$$

$$= \Pr(K_x \geq 5)$$

$$= 5P_x$$

$$= \frac{l_{50}}{l_{75}}$$

$$= \frac{3,914,365}{5,396,081}$$

$$\approx 0.72541$$
8. (6 points) You are given:

   a. \( \ddot{A}_x = 7 \)
   b. \( i = 0.08 \)
   c. \( q_{x+t} = 0.01(t+1) \)

Calculate \( 1000A_{x+2} \).

\[
A_x = 1 - d \cdot \dot{A}_x \\
= 1 - (1 - 1.08^{-1}) \times 7 \\
= 0.48148
\]

\( q_x = 0.01 \)
\( q_{x+1} = 0.02 \)

By Recursion Formula,

\[
A_x = vq_x + vp_x \cdot A_{x+1}
\]

\[
\Rightarrow A_{x+1} = \frac{A_x - vq_x}{vp_x} = \frac{0.48148 - 1.08^{-1} \times 0.01}{1.08^{-1} \times (1 - 0.01)} = 0.51515
\]

\[
A_{x+2} = \frac{A_{x+1} - vq_{x+1}}{vp_{x+1}} = \frac{0.51515 - 1.08^{-1} \times 0.02}{1.08^{-1} \times (1 - 0.02)} \approx 0.54731 \Rightarrow 1000A_{x+2} = 547.31
\]