1. You are given the following empirical distribution of losses:

\[ 300 \ 500 \ 700 \ 800 \ 1000 \ 1400 \]

An insurance policy covering these losses has an upper limit of 1200 and an ordinary deductible of 500. The upper limit is applied first and then the deductible is applied.

Calculate the expected payment per payment for this policy.

**Solution:**

\[
\begin{array}{ccc}
X & X \wedge 1200 & Y^p \\
300 & 300 & \text{Undefined} \\
500 & 500 & \text{Undefined} \\
700 & 700 & 200 \\
800 & 800 & 300 \\
1000 & 1000 & 500 \\
1400 & 1200 & 700 \\
\end{array}
\]

\[
E[Y^p] = \frac{200 + 300 + 500 + 700}{4} = 425
\]
2. $X$ is distributed as a two point mixture distribution. The two distributions are weighted evenly. One distribution is the exponential distribution with $\theta = 1000$. The other distribution is the Weibull distribution with $\theta = 500$ and $\tau = 1$.

Calculate the median.

Solution:

$Y_1 \sim \text{Exponential}(\theta = 100)$ and $Y_2 \sim \text{Weibull}(\theta = 500, \tau = 1)$

\[
F_{Y_1}(x) = 1 - e^{-\frac{x}{1000}} \quad \text{and} \quad F_{Y_2}(x) = 1 - e^{-\frac{x}{500}}
\]

\[
F_X(y) = 0.5[F_{Y_1}(x) + F_{Y_2}(x)] = 0.5 \left[1 - e^{-\frac{x}{1000}} + 1 - e^{-\frac{x}{500}}\right] = 1 - 0.5e^{-\frac{x}{1000}} - 0.5e^{-\frac{x}{500}}
\]

Median $\pi_{0.5} \implies 0.5 = F_X(\pi_{0.5}) = 1 - 0.5e^{-\frac{\pi_{0.5}}{1000}} - 0.5e^{-\frac{\pi_{0.5}}{500}}$

\[\implies 0.5e^{-\frac{\pi_{0.5}}{500}} + 0.5e^{-\frac{\pi_{0.5}}{1000}} = 0.5 \implies e^{-\frac{\pi_{0.5}}{500}} + e^{-\frac{\pi_{0.5}}{1000}} - 1 = 0\]

Let $x = e^{-\frac{\pi_{0.5}}{1000}} \implies x^2 = e^{-\frac{\pi_{0.5}}{1000}} - e^{-\frac{\pi_{0.5}}{500}} = x + x - 1 = 0$

\[x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{(2)(1)} \implies x = 0.6180 \implies e^{-\frac{\pi_{0.5}}{1000}} = 0.6180\]

$\therefore \pi_{0.5} = -1000\ln(0.6180) = 481.2118$
3. For a zero modified negative binomial distribution, you are given:

1. \( p_2^M = 1/25 \)
2. \( p_3^M = p_4^M = \frac{16}{375} \)

Calculate \( E[N] \).

**Solution:**

\[
p_k = \left( a + \frac{b}{k} \right) p_{k-1} \quad \text{for } k = 2, 3, \ldots \quad \text{where } a = \frac{\beta}{\beta + 1} \quad \text{and } b = \frac{(\gamma - 1)\beta}{\beta + 1}
\]

\[
\frac{p_1}{p_2} = \frac{16/375}{1/25} = \frac{16}{15} = a + \frac{b}{3} \quad \text{and} \quad \frac{p_4}{p_3} = 1 = a + \frac{b}{4}
\]

\[\Rightarrow \frac{16}{15} = a + \frac{b}{3} \quad \Rightarrow \frac{1}{15} = \frac{b}{12} \quad \Rightarrow b = 0.8
\]

\[a = 1 - \frac{b}{4} = 1 - 0.2 = 0.8 = \frac{\beta}{\beta + 1} \quad \Rightarrow \beta = 4
\]

\[b = \frac{(\gamma - 1)\beta}{\beta + 1} \quad \Rightarrow 0.8 = (\gamma - 1)(0.8) \quad \Rightarrow \gamma = 2
\]

\[E[N^M] = (1 - p_0^M) (5) E[N^T] \quad \text{so we need } p_0^M \quad \text{but } p_2^M = (1 - p_0^M) p_2^T
\]

\[\Rightarrow \frac{1}{25} = (1 - p_0^M) \frac{\gamma(\gamma + 1)}{2![(\beta + 1)^2 - 1]} \left( \frac{\beta}{\beta + 1} \right)^2 = (1 - p_0^M) \frac{2(2+1)}{2![(4+1)^2 - 1]} \left( \frac{4}{4+1} \right)^2
\]

\[\frac{1}{25} = (1 - p_0^M) \frac{2}{25} \quad \Rightarrow p_0^M = 0.5
\]

\[E[N^M] = (1 - p_0^M) E[N^T] = (1 - 0.5) \frac{\gamma \beta}{1 - (1 + \beta)^{-\gamma}} = 0.5 \frac{(2)(4)}{1 - (5)^{-2}} = \frac{25}{6} = 4.16667
\]
4. Losses for liability claims represented by the random variable \( X \) are distributed as an exponential distribution with a mean of 10,000.

Chen Insurance has discovered that if the company pays a bonus to its agents that the distribution of the losses will change. The losses represented by the random variable \( Y \) is then distributed as an exponential distribution with a mean of 9000.

Chen implements the bonus program where the company pays the claim plus \( B(10,000 - Y) \). A negative bonus cannot be paid.

Determine \( B \) so that the Company shares 50% of the expected savings in claims with the agent. The expected savings = \( E[X] - E[Y] \).

Solution:

\[
E[X] = 10,000 \quad \text{and} \quad E[Y] = 9000 \implies \text{Expected Savings} = E[X] - E[Y] = 1000
\]

\[
E[\text{Bonus}] = E[B(10,000 - (Y \land 10,000))] = (0.5)(\text{Expected Savings}) = 500
\]

\[
B[10,000 - E(Y \land 10,000)] = 500
\]

\[
B \left[ 10,000 - \theta \left(1 - e^{-\frac{10,000}{\theta}}\right)\right] = 500
\]

\[
B \left[ 10,000 - 9000 \left(1 - e^{-\frac{10,000}{9000}}\right)\right] = 500
\]

\[
B[10,000 - 6037.2631] = 500
\]

\[
B = \frac{500}{10,000 - 6037.2631} = 0.126175
\]
5. During 2013, the losses incurred under on dental insurance are distributed as a Pareto distribution with $\theta = 500$ and $\alpha = 6$.

During 2013, Amstutz Assurance Company sold a dental policy with an upper limit of 150.

Losses in 2014 are expected to increase uniformly by 12.5%. During 2014, Amstutz sells a dental policy with a franchise deductible of $D^F$ and no upper limit.

The $\frac{E[Y^F]}{E[Y^L]}$ for 2014 = 2

Calculate $D^F$.

**Solution:**

$X_{13} \sim \text{Pareto}(\theta = 500, \alpha = 6) \Rightarrow E[Y_{13}^F] = E[Y_{13}^L]$ since there is no deductible =

$$E[X_{13} \land 150] = \frac{\theta}{\alpha - 1} \left( 1 - \left( \frac{\theta}{150 + \theta} \right)^{\alpha - 1} \right) = \frac{500}{6 - 1} \left( 1 - \left( \frac{500}{150 + 500} \right)^{6 - 1} \right) = 73.06709257$$

$X_{14} \sim \text{Pareto}(\theta = 562.5, \alpha = 6)$ since $\theta$ is a scale parameter and $(500)(1.125)=562.50$

$$E[Y_{14}^F] = \frac{E[Y_{14}^L]}{1 - F(D^F)} = \frac{E[X_{14}] - E[X_{14} \land D^F] + (D^F)(1 - F(D^F))}{1 - F(D^F)} = \frac{\theta}{\alpha - 1} - \frac{\theta}{\alpha - 1} \left[ 1 - \left( \frac{\theta}{\theta + D^F} \right)^{\alpha - 1} \right] + D^F$$

$$= \frac{\theta}{\alpha - 1} - \frac{\theta}{\alpha - 1} \left( \frac{\theta}{\theta + D^F} \right)^{\alpha - 1} + D^F = \frac{\theta}{\alpha - 1} + D^F = \frac{\theta + D^F}{\alpha - 1} + D^F = \frac{562.5}{5} + D^F$$

$$\frac{E[Y_{14}^F]}{E[Y_{13}^F]} = 2 \Rightarrow \frac{562.5 + D^F}{73.06709257} = 112.50 + 1.2D^F = 2 \Rightarrow D^F = \frac{(2)(73.06709257) - 112.50}{1.2} = 28.03$$
6. Goh Guaranty Insurance Company provides coverage for iPhone damage. Claims can take on amounts of 50, 100, and 200.

The number of claims received by Goh follows a Poisson distribution with a mean of 48 claims per 24 hour day. The claims are distributed as follows:
   a. 50% of claims are for an amount of 50;
   b. 30% of claims are for an amount of 100; and
   c. 20% of claims are for an amount of 200.

Calculate the probability that during a one hour period, the total amount of claims received will be less than or equal to 100.

Solution:

48 claims in 24 hours means 2 claims per hour.

\[ X_{50} \sim \text{Poisson}(\lambda = (2)(0.5) = 1) \]
\[ X_{100} \sim \text{Poisson}(\lambda = (2)(0.3) = 0.6) \]
\[ X_{200} \sim \text{Poisson}(\lambda = (2)(0.2) = 0.4) \]

\[
\Pr(\text{Total} \leq 100) = \Pr(X_{50} = 0; X_{100} = 0; X_{200} = 0) + \Pr(X_{50} = 1; X_{100} = 0; X_{200} = 0) \\
\quad + \Pr(X_{50} = 2; X_{100} = 0; X_{200} = 0) + \Pr(X_{50} = 0; X_{100} = 1; X_{200} = 0)
\]

\[
= e^{-1}e^{-0.6}e^{-0.4} + \frac{(e^{-1})(1^1)}{1!}e^{-0.6}e^{-0.4} + \frac{(e^{-1})(1^2)}{2!}e^{-0.6}e^{-0.4} + e^{-1}\frac{(e^{-0.6})(0.6)^1}{1!}e^{-0.4} = 
\]

\[
e^{-2}(1+1+0.5+0.6) = 0.41954
\]
7. You are given that \( f(x) = \frac{x^2}{9000} \) for \( 0 \leq x \leq 30 \).

If \( \text{VaR}_p(x) = 21 \), calculate \( \text{TVaR}_p(x) \).

Solution:

\[
F_x(x) = \int_0^x f(s) \, ds = \int_0^x \frac{s^2}{9000} \, ds = \left[ \frac{1}{27,000} s^3 \right]^x_0 = \frac{x^3}{27,000} \quad \text{for } 0 \leq x \leq 30
\]

\[
\text{VaR}_p = 21 \implies p = F(21) = \frac{21^3}{27,000} = 0.343
\]

\[
\text{TVaR}_p = E[X \mid X > \text{VaR}_p] = \int_{21}^{30} x f(x) \, dx = \frac{\int_{21}^{30} x^2 \, dx}{1 - p} = \frac{\int_{21}^{30} x^3 \, dx}{0.657} = \frac{30^4 - 21^4}{0.657} = 26.02397
\]
8. You are given that claims are distributed as a gamma distribution with $\alpha = 3$ and $\theta$. You are also given that $\theta$ is distributed uniformly between 1000 and 5000.

Chellberg Casualty Company wants to establish a premium equal to the expected claims plus one standard deviation. Calculate the premium that Chellberg should charge.

Solution:

$$E[E[X \mid \theta]] = E[\alpha \theta] = 3E[\theta] = 3 \frac{1000 + 5000}{2} = 9000$$

$$Var(X) = E[Var(X \mid \theta)] + Var(E[X \mid \theta]) = E[\alpha \theta^2] + Var(\alpha \theta) =$$

$$3E[\theta^2] + 3^2 Var(\theta) = 3 \int_{1000}^{5000} \theta^2 \frac{1}{4000} d\theta + 9 \frac{(5000 - 1000)^2}{12} =$$

$$3 \frac{\theta^3}{12,000} \bigg|_{1000}^{5000} + 9(1333333.333333) = 31,000,000 + 12,000,000 = 43,000,000$$

$$Premium = E[X] + \sqrt{Var(X)} = 9000 + \sqrt{43,000,000} = 15,557.44$$
9. Dai Disability Company has 10,000 disability policies in force. Each policy is independent of the other policies. The amount of claims for each policy is distributed as an Inverse Gamma distribution with $\theta = 2000$ and $\alpha = 6$.

Using a normal approximation, calculate the probability that the total claims in a year will be less than 4.025 million.

**Solution:**

\[
E[X] = \frac{\theta}{\alpha - 1} = \frac{2000}{6 - 1} = 400
\]

\[
E[X^2] = \frac{\theta^2}{(\alpha - 1)(\alpha - 2)} = \frac{(2000)^2}{(6 - 1)(6 - 2)} = 200,000
\]

\[
Var(X) = E[X^2] - (E[X])^2 = 200,000 - (400)^2 = 40,000
\]

\[
S = \sum_{i=1}^{10,000} X_i \implies E[S] = 10,000E[X] = (10,000)(400) = 4,000,000
\]

\[
Var(S) = 10,000 Var(X) = (10,000)(40,000) = 400,000,000
\]

\[
Pr[S < 4,025,000] = \Pr \left[ z < \frac{4,025,000 - 4,000,000}{\sqrt{400,000,000}} \right]
\]

\[
Pr(z < 1.25) = 0.8944
\]