Related Rates Word Problems Method:
1. Read problem carefully several times to understand what is asked.
2. Draw a picture (if possible) and label.
3. Write down the given rate; write down the desired rate.
4. Find an equation relating the variables.
5. Use Chain Rule to differentiate equation w.r.t to time and solve for desired rate.

The Linear Approximation (or tangent line approximation) to a function \( f(x) \) at \( x = a \) is the function \( L(x) = f(a) + f'(a)(x - a) \); Approximation formula \( f(x) \approx f(a) + f'(a)(x - a) \) for \( x \) near \( a \); if \( y = f(x) \), the differential of \( y \) is \( dy = f'(x)dx \).

Definitions of absolute maximum, absolute minimum, local/relative maximium, and local/relative minimum; \( c \) is a critical number of \( f \) if \( c \) is in the domain of \( f \) and either \( f'(c) = 0 \) or \( f'(c) \) DNE.

Extreme Value Theorem: If \( f(x) \) is continuous on a closed interval \([a, b]\), then \( f \) always has an absolute maximum value and an absolute minimum value on \([a, b]\).

Method to Find Absolute Max/Min of \( f(x) \) over Closed Interval \([a, b]\):
(i) Find all admissible critical numbers in \((a, b)\);
(ii) Find endpoints of interval;
(iii) Make table of values of \( f(x) \) at the points found in (i) and (ii).

The largest value = abs max value of \( f \) and the smallest value = abs min value of \( f \).

Rolle’s Theorem: If \( f(x) \) is continuous on \([a, b]\) and differentiable on \((a, b)\), and \( f(a) = f(b) \), then \( f'(c) = 0 \) for some \( c \in (a, b) \).

Mean Value Theorem: If \( f(x) \) is continuous on \([a, b]\) and differentiable on \((a, b)\), then there is a number \( c \), where \( a < c < b \), such that \( \frac{f(b) - f(a)}{b - a} = f'(c) \):

\[ y = f(x) \]

i.e., \( f(b) - f(a) = f'(c)(b - a) \)

If something about \( f' \) is known, then something about the sizes of \( f(a) \) and \( f(b) \) can be found.

Fact: (Useful for integration theory later)
(a) If \( f'(x) = 0 \) for all \( x \in I \), then \( f(x) = C \) for all \( x \in I \).
(b) If \( f'(x) = g'(x) \) for all \( x \in I \), then \( f(x) = g(x) + C \) for all \( x \in I \).
Increasing functions: $f'(x) > 0 \implies f \nearrow$; decreasing functions: $f'(x) < 0 \implies f \searrow$.

**Increasing/Decreasing**

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f'(x)$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>-</td>
<td>↓</td>
</tr>
<tr>
<td>-3</td>
<td>-</td>
<td>↓</td>
</tr>
<tr>
<td>-2</td>
<td>-</td>
<td>↓</td>
</tr>
<tr>
<td>-1</td>
<td>+</td>
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<tr>
<td>0</td>
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<td>1</td>
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<td>2</td>
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<tr>
<td>3</td>
<td>+</td>
<td>↑</td>
</tr>
<tr>
<td>4</td>
<td>-</td>
<td>↓</td>
</tr>
</tbody>
</table>

(local min) (local max) (critical pt) (not local max or min)

**First Derivative Test**: Suppose $c$ is a critical number of a continuous function $f$.

(a) If $f'$ changes from $+$ to $-$ at $c \implies f$ has local max at $c$
(b) If $f'$ changes from $-$ to $+$ at $c \implies f$ has local min at $c$
(c) If $f'$ does not change sign at $c \implies f$ has neither local max nor local min at $c$

(Displaying this information on a number line is much more efficient, see above figure.)

**Concave Up/Down**

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f''(x)$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>+</td>
<td>↓</td>
</tr>
<tr>
<td>-3</td>
<td>+</td>
<td>↓</td>
</tr>
<tr>
<td>-2</td>
<td>+</td>
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</tr>
<tr>
<td>-1</td>
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<tr>
<td>0</td>
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<td>↑</td>
</tr>
<tr>
<td>4</td>
<td>-</td>
<td>↓</td>
</tr>
</tbody>
</table>

(inflection pt) (inflection pt) (not inflection pt)

(Displaying this information on a number line is much more efficient, see above figure.)

**Second Derivative Test**: Suppose $f''$ is continuous near critical number $c$ and $f'(c) = 0$.

(a) If $f''(c) > 0 \implies f$ has a local min at $c$.
(b) If $f''(c) < 0 \implies f$ has a local max at $c$.

**Note**: If $f''(c) = 0$, then 2nd Derivative Test cannot be used, so then use 1st Derivative Test.

**Indeterminate Forms**:

(a) Indeterminate Form (Types): $0 \cdot \infty$, $\infty - \infty$, $0^0$, $\infty^0$, $1^\infty$

(b) L’Hopital’s Rule: Let $f$ and $g$ be differentiable and $g'(x) \neq 0$ on an open interval $I$ containing $a$ (except possibly at $a$). If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ [Type $0/0$]; or if $\lim_{x \to a} f(x) = \infty$ (or $-\infty$) and $\lim_{x \to a} g(x) = \infty$ (or $-\infty$) [Type $\infty/\infty$], then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right side exists or is infinite.

Use algebra to convert the different Indeterminate Forms in (a) into expressions where the above formula can be used.

**Important Remark**: L’Hopital’s Rule is also valid for one-sided limits, $x \to a^-$, $x \to a^+$, and also for limits when $x \to \infty$ or $x \to -\infty$. 
14 Curve Sketching Guidelines:
(a) Domain of \( f \)
(b) Intercepts (if any)
(c) Symmetry:
   \( f(-x) = f(x) \) for even function;
   \( f(-x) = -f(x) \) for odd function;
   \( f(x + p) = f(x) \) for periodic function
(d) Asymptotes:
   \( x = a \) is a Vertical Asymptote: if either \( \lim_{x \to a^-} f(x) \) or \( \lim_{x \to a^+} f(x) \) is infinite
   \( y = L \) is a Horizontal Asymptote: if either \( \lim_{x \to \infty} f(x) = L \) or \( \lim_{x \to -\infty} f(x) = L \).
(e) Intervals: where \( f \) is increasing \( \uparrow \) and decreasing \( \downarrow \); local max and local min
(f) Intervals: where \( f \) is concave up \( \cup \) and concave down \( \cap \); inflection points

15 Optimization (Max/Min) Word Problems Method:
1. Read problem carefully several times.
2. Draw a picture (if possible) and label it.
3. Introduce notation for the quantity, say \( Q \), to be extremized as a function of one or more variables.
4. Use information given in problem to express \( Q \) as a function of only one variable, say \( x \). Write the domain of \( Q \).
5. Use Max/Min methods to determine the absolute maximum value of \( Q \) or the absolute minimum of \( Q \), whichever was asked for in problem.

16 Integration Theory:
(a) \( F(x) \) is an antiderivative of \( f(x) \), if \( F'(x) = f(x) \).
(b) Definite Integral \( \int_a^b f(x) \, dx \) is a number; gives the net area under a curve \( y = f(x) \) when \( a \leq x \leq b \); also gives net distance traveled by particle with velocity \( y = f(x) \) from time \( x = a \) to \( x = b \); many other applications (take Calculus II).
(c) Properties of Definite Integrals.
(d) FUNDAMENTAL THEOREM OF CALCULUS:
   I If \( f(x) \) is continuous on \( [a, b] \) and \( g(x) = \int_a^x f(t) \, dt \implies g'(x) = f(x) \).
   \[ \frac{d}{dx} \left( \int_a^x f(t) \, dt \right) = f(x) \] Leibniz’ Rule
   II If \( F(x) \) is any antiderivative of \( f(x) \) \( \implies \int_a^b f(x) \, dx = F(x) \bigg|_{x=a}^{x=b} = F(b) - F(a) \).
   \[ \int_a^b F'(x) \, dx = F(b) - F(a) \] Net Change Rule
(e) Indefinite Integral \( \int f(x) \, dx \) is a function.
   Recall that \( \int f(x) \, dx = F(x) \) means \( F'(x) = f(x) \), i.e. the indefinite integral \( \int f(x) \, dx \) is simply the most general antiderivative of \( f(x) \).
(f) Generalized Leibniz’ Rule:
   \[ \frac{d}{dx} \left( \int_a^{u(x)} f(t) \, dt \right) = f(u(x)) \frac{du(x)}{dx} \]
(g) **Substitution Rule** (Indefinite Integrals): \[ \int f(g(x))g'(x)\,dx = \int f(u)\,du, \quad u = g(x). \]

(h) **Substitution Rule** (Definite Integrals): \[ \int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du, \quad u = g(x). \]

**Basic Table of Indefinite Integrals**

(1) \( \int k\,dx = kx + C \)

(2) \( \int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \)

(3) \( \int \frac{1}{x}\,dx = \ln|x| + C \)

(4) \( \int e^x\,dx = e^x + C \)

(5) \( \int \cos x\,dx = \sin x + C \)

(6) \( \int \sin x\,dx = -\cos x + C \)

(7) \( \int \sec^2 x\,dx = \tan x + C \)

(8) \( \int \sec x \tan x\,dx = \sec x + C \)

(9) \( \int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C \)

(10) \( \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C \)

(11) \( \int \sinh x\,dx = \cosh x + C \)

(12) \( \int \cosh x\,dx = \sinh x + C \)

(13) \( \int a^x\,dx = \frac{a^x}{\ln a} + C \)