1. Double integrals; Midpoint Rule for rectangle: 
\[
\iint_R f(x, y) \, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j) \Delta A;
\]

2. Type I region \( D \): \( \{ g_1(x) \leq y \leq g_2(x) \} \); Type II region \( D \): \( \{ h_1(y) \leq x \leq h_2(y) \} \);

iterated integrals over Type I and II regions:
\[
\int \int_D f(x, y) \, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx \quad \text{and} \quad \int \int_D f(x, y) \, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \, dy,
\]
respectively; Reversing Order of Integration (regions that are both Type I and Type II); properties of double integrals.

3. Integral inequalities: 
\[
mA \leq \iint_D f(x, y) \, dA \leq MA, \quad \text{where} \quad A = \text{area of } D \text{ and } m \leq f(x, y) \leq M \text{ on } D.
\]

4. Change of Variables Formula in Polar Coordinates: if \( D \): \( \{ h_1(\theta) \leq r \leq h_2(\theta) \} \), then
\[
\int \int_D f(x, y) \, dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta.
\]

5. Applications of double integrals:

(a) Area of region \( D \) is \( A(D) = \iint_D \, dA \)

(b) Volume of solid under graph of \( z = f(x, y) \), where \( f(x, y) \geq 0 \), is \( V = \iiint_D f(x, y) \, dV \)

(c) Mass of \( D \) is \( m = \iiint_D \rho(x, y) \, dA \), where \( \rho(x, y) = \text{density (per unit area)} \); sometimes write
\[
m = \iiint_D \, dm, \quad \text{where } dm = \rho(x, y) \, dA.
\]

(d) Moment about the \( x \)-axis \( M_x = \iiint_D y \, \rho(x, y) \, dA \); moment about the \( y \)-axis \( M_y = \iiint_D x \, \rho(x, y) \, dA \).

(e) Center of mass \( (\bar{x}, \bar{y}) \), where \( \bar{x} = \frac{M_y}{m} = \frac{\iiint_D x \, \rho(x, y) \, dA}{\iiint_D \rho(x, y) \, dA} \), \( \bar{y} = \frac{M_x}{m} = \frac{\iiint_D y \, \rho(x, y) \, dA}{\iiint_D \rho(x, y) \, dA} \)

Remark: centroid = center of mass when density is constant (this is useful).

6. Elementary solids \( E \subset \mathbb{R}^3 \) of Type 1, Type 2, Type 3; triple integrals over solids \( E \):
\[
\iiint_E f(x, y, z) \, dV = \iiint_D \int_{u(x,y)}^{v(x,y)} f(x, y, z) \, dz \, dA \quad \text{for } E = \{(x, y) \in D, \ u(x, y) \leq z \leq v(x, y)\};
\]

volume of solid \( E \) is \( V(E) = \iiint_E \, dV \); applications of triple integrals, mass of a solid, moments about the coordinate planes \( M_{xy}, M_{xz}, M_{yz} \), center of mass of a solid \( (\bar{x}, \bar{y}, \bar{z}) \).
7. **Cylindrical Coordinates** \((r, \theta, z)\):

From CC to RC:
\[
\begin{align*}
    x &= r \cos \theta \\
y &= r \sin \theta \\
z &= z
\end{align*}
\]

Going from RC to CC use \(x^2 + y^2 = r^2\) and \(\tan \theta = \frac{y}{x}\) (make sure \(\theta\) is in correct quadrant).

8. **Spherical Coordinates** \((\rho, \theta, \phi)\), where \(0 \leq \phi \leq \pi\):

From SC to RC:
\[
\begin{align*}
x &= (\rho \sin \phi) \cos \theta \\
y &= (\rho \sin \phi) \sin \theta \\
z &= \rho \cos \phi
\end{align*}
\]

Going from RC to SC use \(x^2 + y^2 + z^2 = \rho^2\), \(\tan \theta = \frac{y}{x}\) and \(\cos \phi = \frac{z}{\rho}\).

9. Triple integrals in Cylindrical Coordinates:
\[
\iiint_{E} f(x, y, z) \, dV = \iiint_{E} f(r \cos \theta, r \sin \theta, z) \, r \, dz \, dr \, d\theta
\]

10. Triple integrals in Spherical Coordinates:
\[
\iiint_{E} f(x, y, z) \, dV = \iiint_{E} f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \, \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
\]

11. Vector fields on \(\mathbb{R}^2\) and \(\mathbb{R}^3\): \(\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle\) and \(\vec{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle\);
\(\vec{F}\) is a conservative vector field if \(\vec{F} = \nabla f\), for some real-valued function \(f\).
12. Line integral of a function $f(x, y)$ along $C$, parameterized by $x = x(t), \ y = y(t)$ and $a \leq t \leq b$, is

$$
\int_C f(x, y) \, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt.
$$

(independent of orientation of $C$, other properties and applications of line integrals of $f$)

**Remarks:**

(a) $\int_C f(x, y) \, ds$ is sometimes called the “line integral of $f$ with respect to arc length”

(b) $\int_C f(x, y) \, dx = \int_a^b f(x(t), y(t)) \, x'(t) \, dt$

(c) $\int_C f(x, y) \, dy = \int_a^b f(x(t), y(t)) \, y'(t) \, dt$

13. Line integral of vector field $\vec{F}(x, y)$ along $C$, parameterized by $\vec{r}(t)$ and $a \leq t \leq b$, is given by

$$
\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt.
$$

(depends on orientation of $C$, other properties and applications of line integrals of $f$)

14. Connection between line integral of vector fields and line integral of functions:

$$
\int_C \vec{F} \cdot d\vec{r} = \int_C (\vec{F} \cdot \vec{T}) \, ds
$$

where $\vec{T}$ is the unit tangent vector to the curve $C$.

15. If $\vec{F}(x, y) = P(x, y) \vec{i} + Q(x, y) \vec{j}$, then $\int_C \vec{F} \cdot d\vec{r} = \int_C P(x, y) \, dx + Q(x, y) \, dy$; Work = $\int_C \vec{F} \cdot d\vec{r}$.

16. **Fundamental Theorem of Calculus for Line Integrals:**

$$
\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))
$$

17. A vector field $\vec{F}(x, y) = P(x, y) \vec{i} + Q(x, y) \vec{j}$ is conservative (i.e. $\vec{F} = \nabla f$) if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$; how to determine a potential function $f$ if $\vec{F}(\vec{x}) = \nabla f(\vec{x})$.

18. **Green’s Theorem:**

$$
\int_C P(x, y) \, dx + Q(x, y) \, dy = \int_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA \quad (C = \text{boundary of } D):
$$
19. **Del Operator**: \( \nabla = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} \); if \( \mathbf{F}(x, y, z) = P(x, y, z) \mathbf{i} + Q(x, y, z) \mathbf{j} + R(x, y, z) \mathbf{k} \), then

\[
\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \quad \text{and} \quad \text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}
\]

Properties of curl and divergence:

(i) If \( \text{curl } \mathbf{F} = \mathbf{0} \), then \( \mathbf{F} \) is a conservative vector field (i.e., \( \mathbf{F}(\mathbf{x}) = \nabla f(\mathbf{x}) \)).

(ii) If \( \text{curl } \mathbf{F} = \mathbf{0} \), then \( \mathbf{F} \) is **irrotational**; if \( \text{div } \mathbf{F} = 0 \), then \( \mathbf{F} \) is **incompressible**.

(iii) **Laplace’s Equation**: \( \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \).