MA 266
Review Topics - Exam #1

(1) Solving Special Types of First Order Equations

I. First Order Linear Equation (FOL):

\[ \frac{dy}{dt} + p(t)y = g(t) \]

Method of Solution: \[ y = \frac{1}{\mu(t)} \left[ \int \mu(t)g(t) \, dt + C \right], \quad \text{where} \quad \mu(t) = e^{\int p(t) \, dt} \]

II. Separable Equation (SEP):

\[ \frac{dy}{dx} = f(x) \, g(y) \]

Method of Solution: \[ \int \frac{1}{g(y)} \, dy = \int f(x) \, dx \]

(The solution is usually given implicitly by the above formula. You may get additional solutions from \( g(y) = 0 \). You must check to see if there are extra solutions.)

III. Homogeneous Equation (HOME):

\[ \frac{dy}{dx} = f(x, y), \quad \text{where} \quad f(tx, ty) = f(x, y) \]

Method of Solution: Let \( \frac{y}{x} = v \), so that \( y = vx \) and \( \frac{dy}{dx} = x \frac{dv}{dx} + v \).

Substitute these into \( \frac{dy}{dx} = f(x, y) \) to obtain a Separable Equation.

IV. Exact Equation (EXE):

\[ M(x, y) \, dx + N(x, y) \, dy = 0, \quad \text{where} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \]

Method of Solution: Implicit solution given by \( \psi(x, y) = C \), where:

\[
\begin{align*}
\frac{\partial \psi}{\partial x} &= M(x, y) \implies \psi = \int M(x, y) \, dx + h(y) \\
\downarrow \\
\frac{\partial \psi}{\partial y} &= N(x, y) = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y} \left( \int M(x, y) \, dx + h(y) \right)
\end{align*}
\]
(2) **Direction Fields.** A solution \( y = \phi(t) \) to the d.e. \( \frac{dy}{dt} = f(t, y) \) has slope \( f(t, y) \) at the point \( (t, y) \). The *direction field* (or *slope field*) of the d.e. indicates the slope of solutions at various points \( (t, y) \). The direction field may be used to give qualitative information about the behavior of solutions as \( t \to \infty \) (or \( t \to -\infty \), or \( t \to 0 \), etc). Direction fields may also be used to approximate the interval where a solution through a point \( (t_0, y_0) \) is defined.

(3) **Applications of 1st Order Equations.**

I. **Mixing Problems:** \( Q(t) = \) amount of substance in solution at time \( t \)

\[
\frac{dQ}{dt} = IN - OUT = r_i c_i - r_o c_o
\]

II. **Exponential Growth/Decay:** \( Q(t) = \) quantity present at time \( t \)

\[
\frac{dQ}{dt} = r Q
\]

III. **Newton’s Law of Cooling:** \( T(t) = \) temperature at time \( t \)

\[
\frac{dT}{dt} = k (T - T_a)
\]

\( (T_a = \text{temperature of surrounding environment}) \)

IV. **Falling & Rising Objects:** You should be able to set up and solve simple problems using Newton’s 2\(^{nd}\) Law: \( F = m \frac{dv}{dt} \). Near the surface of the Earth, the force due to gravity is the weight of the object \( F_g = mg \). For rising objects, let \( v(t) \) be positive *upward*; for falling bodies let \( v(t) \) be positive *downward*.

[Diagrams of rising and falling bodies with forces labeled]
4 Existence and Uniqueness of Solutions.

a) **THEOREM (First Order Linear).** If \( p(t) \) and \( g(t) \) are continuous on an interval \( \alpha < t < \beta \) containing \( t_0 \), then the IVP \[
\begin{cases}
y' + p(t)y = g(t) \\
y(t_0) = y_0
\end{cases}
\] has a unique solution \( y = \phi(t) \) on the interval \( \alpha < t < \beta \), for any \( y_0 \).

*Note:* The largest such open interval containing \( t_0 \) is where the solution \( y = \phi(t) \) is guaranteed to exist.

(b) **THEOREM (First Order Nonlinear).** If \( f(t, y) \) and \( \frac{\partial f}{\partial y} \) are continuous in some rectangle \( \mathbf{R} \): \( \alpha < t < \beta \), and \( \gamma < y < \delta \) and \((t_0, y_0)\) lies inside the rectangle \( \mathbf{R} \), then the IVP \[
\begin{cases}
y' = f(t, y) \\
y(t_0) = y_0
\end{cases}
\] has a unique solution on the interval \( t_0 - h < t < t_0 + h \), for some number \( h > 0 \).

*Note:* The number \( h \) is not easy to find. The interval containing \( t_0 \) where solution exists can be estimated by looking at the direction field of the differential equation. To determine the exact interval, you must solve the IVP explicitly for \( y \).

5 Autonomous Equations: Equations of the form

\[
\frac{dy}{dt} = F(y) \tag{*}
\]

are said to be autonomous since \( \frac{dy}{dt} \) does not depend on the independent variable \( t \). Such equations can have constant solutions (i.e., \( y = K \)) which are called equilibrium solutions. These solutions are found by solving \( F(y) = 0 \). (These are also called critical points.) You should be able to find all equilibrium solutions to the autonomous d.e. \((*)\) and sketch non-equilibrium solutions using the phase line of the differential equation \((*)\). You should also be able to classify the stability of the equilibrium solutions as follows:

a) **Asymptotically Stable** - Solutions which start near \( y = K \) will always approach \( y = K \) as \( t \to \infty \):

![Asymptotically Stable](image)

(b) **Asymptotically Unstable** - Solutions which start near \( y = K \) does not always approach \( y = K \) as \( t \to \infty \):

![Asymptotically Unstable](image)
(c) **Semistable** - This is a special type of *unstable* solution. In this case solutions on one side of \( y = K \) will approach \( y = K \) as \( t \to \infty \), while solutions on the other side of \( y = K \) will approach something else:

![Diagram showing semistable behavior](image)

**Remark.** To sketch non-equilibrium solutions of \((*)\), you do not necessarily need direction fields, you can use ordinary calculus. Since \( \frac{dy}{dt} = F(y) \), the graph of \( F(y) \) vs \( y \) will determine where the solution \( y = \phi(t) \) is increasing \((F(y) > 0)\) or decreasing \((F(y) < 0)\). By the Chain Rule, \( \frac{d^2y}{dt^2} = \frac{dF(y)}{dy} F(y) \), hence a graph of \( \frac{dF}{dy} \) will determine where the solution \( y = \phi(t) \) is concave up \((F'F > 0)\) or concave down \((F'F < 0)\).

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**Practice Problems**

1. Determine the order of each of these differential equations; also state whether the equation is *linear* or *nonlinear*:
   - (a) \( yy' + x = 1 \)
   - (b) \( xy' + y = 1 \)
   - (c) \( (y')^3 + ty = 1 \)
   - (d) \( y'' + \sqrt{t}y = 1 \)

2. (a) Which of the functions \( y_1(t) = t \) and \( y_2(t) = -t \) is/are solutions of the IVP \( y' = t, \ y(0) = 0 \)?
   - (b) Which of the functions \( y_1(t) = t \) and \( y_2(t) = -t \) are is/solutions of the IVP \( y' = t, \ y(1) = 1 \)?

3. For what value(s) of \( r \) is \( y = e^{rx} \) a solution of \( y'' - 5y' + 6y = 0 \)?

4. (a) Show that \( y = x^3 \) is a solution of the initial value problem \( y' = 3y^{2/3}, \ y(0) = 0 \).
   - (b) Find a different solution of the initial value problem.

5. Find an explicit solution of the initial value problem \( x^2y' = y^2, \ y(1) = \frac{1}{2} \). Indicate the interval in which the solution is valid.

6. (a) Find an implicit solution of the initial value problem \( y' = \frac{2x}{2y + 1}, \ y(0) = 0 \).
   - (b) Find an explicit solution of the initial value problem \( y' = \frac{2x}{2y + 1}, \ y(0) = 0 \).

7. For what value(s) of \( a \) is the solution of the IVP \( y' - y + 2e^{-t} = 0, \ y(0) = a \) bounded on the interval \( t \geq 0 \)?

8. Determine whether each of the following differential equations is linear, separable, homogeneous, and/or exact or none of these. (a) \( 2x + y + (x + 3y) \frac{dy}{dx} = 0 \)
   - (b) \( x + 3y + (2x + y) \frac{dy}{dx} = 0 \)
   - (c) \( (x + 3y + 1)dx + (2x + y + 1)dy = 0 \)
   - (d) \( 2xy + 1 + (x^2 + 1) \frac{dy}{dx} = 0 \)
   - (e) \( (y^2 + 1)dy + (x^2 + 1)dx = 0 \)
9. Find implicit solutions to
   (a) \( x^2 + y^2 - 2xyy' = 0 \)
   (b) \((1 + y^2)dx - 2xydy = 0\)
   (c) \(x + y^2 + 2xyy' = 0\)

10. Find an implicit form of the general solution of the differential equation \( \frac{dy}{dx} = \frac{x^2 + y^2}{xy} \).

11. Find an implicit solution of the IVP \( 2xy + 1 + (x^2 + 2y)\frac{dy}{dx} = 0, \ y(1) = -1 \).

12. If \( xy' + (x + 1)y = 2xe^{-x} \) and \( y(1) = 0 \), then \( y(2) = ? \)

13. Use the given direction field to sketch the solution of the corresponding initial value problem \( y' = f(t, y) \), \( y(t_0) = y_0 \) for the indicated initial value \((t_0, y_0)\) :
   (a) \((0, 0)\)  
   (b) \((0, 2)\)  
   (c) \((-1, 3)\)  
   (d) \((0, 4)\)  

14. For each of the initial value problems determine the largest interval for which a unique solution is guaranteed:
   (a) \( y' - \frac{2}{t}y = \frac{1}{t}, \ y(1) = 0 \)  
   (b) \( y' + (\tan t)y = \sec t, \ y(0) = 0 \)  
   (c) \( y' + \frac{x}{x^2 - 9}y = \frac{1}{x - 2}, \ y(0) = 1 \)  
   (d) \((x + 4)y' - xy = \frac{1}{x}, \ y(-2) = 1 \)  

15. For each of the initial value problems determine all initial points \((t_0, y_0)\) for which a unique solution is guaranteed in some interval \( t_0 - h < t < t_0 + h \):
   (a) \( y' = t^2 + y^2, \ y(t_0) = y_0 \)  
   (b) \( y' = t/y, \ y(t_0) = y_0 \)  
   (c) \( y' = \sqrt{t^2 + y^2}, \ y(t_0) = y_0 \)  
   (d) \( y' = t^{1/3} + y^{1/3}, \ y(t_0) = y_0 \)  
   (e) \( y' = \frac{\sqrt{1 - y^2}}{t - 2}, \ y(t_0) = y_0 \)
16. Find the explicit solution of the initial value problem \( y' = y^2 - 1, \ y(0) = -2 \). Where is this solution defined?

17. Suppose \( y' \) is proportional to \( y \), \( y(0) = 4 \), and \( y(2) = 2 \). Set up and solve an initial value problem that gives \( y \) in terms of \( t \). For what value of \( t \) does \( y(t) = 3 \)?

18. A thermometer reads 36° when it is moved into a 70° room. Five minutes later the thermometer reads 50°. Set up and solve an initial value problem that gives the thermometer reading \( t \) minutes after it is moved into the room. What will it read ten minutes after it is moved into the room?

19. At time \( t = 0 \) a 500 gallon tank contains 40 pounds of salt mixed in 100 gallons of water. A solution that contains 3 lb of salt per gallon of solution is then pumped into the tank at a constant rate of 5 gal/min. The well-stirred mixture flows out of the tank at the rate of 3 gal/min. Set up and solve an initial value problem that gives the amount of salt in the tank after \( t \) minutes. What is the concentration of salt in the tank at the time the tank becomes full?

20. A huge 300 gallon radiator is full of a 60% antifreeze solution. Pure water is poured in at a rate of 5 gal/min and the stirred mixture is drained at the same rate. How long do we pour water into the radiator to get a 50% antifreeze solution?

21. Set up and solve an initial value problem that gives the vertical velocity of a 128-lb parachutist \( t \) seconds after she jumps from an airplane that is flying horizontally at an altitude of 5000 feet. Assume that air resistance is eight times the speed and ignore horizontal motion and downward direction is positive.

22. Consider the differential equation \( \frac{dy}{dt} = y(y - 2) \).

(a) What are the equilibrium solutions?

(b) Which equilibrium solutions are stable/unstable?

(c) Sketch the graph of the solution of the differential equation for \( t \geq 0 \) with each of the initial values \( y(0) = -2/3, \ y(0) = 0, \ y(0) = 2/3, \ y(0) = 4/3, \ y(0) = 2, \ y(0) = 8/3 \).

(d) Find the explicit solution of the initial value problem \( \frac{dy}{dt} = y(y - 2), y(0) = y_0 \).

(e) For what values of \( t \) is the solution in (d) valid?

23. Consider the differential equation \( \frac{dy}{dt} = F(y) \), where the graph of \( F(y) \) is indicated below.

\[ w = F(y) \]

(a) What are the equilibrium solutions?

(b) Which equilibrium solutions are stable?

(c) Sketch some solutions to \( \frac{dy}{dt} = F(y) \).
(1) (a) 1st order nonlinear  (b) 1st order linear  (c) 1st order nonlinear  (d) 3rd order linear  
(2) (a) \( y_1 \) and \( y_2 \)  (b) \( y_1 \) only  
(3) \( r = 2, r = 3 \)  
(4) (a) \( y' = 3x^2 = 3(x^3)^{2/3} = 3y^{2/3}; 0 = 3(0)^{2/3} \)  (b) \( y \equiv 0 \)  
(5) \( y = \frac{x}{x+1}, \ x > -1 \)  
(6) (a) \( y^2 + y = x^2 \)  (b) \( y = \frac{-1 + \sqrt{4x^2 + 1}}{2} \)  
(7) \( a = 1 \)  
(8) (a) HOME and EXE  (b) HOME 
(c) none of these types  (d) FOL and EXE  (e) SEP and EXE  
(9) (a) (HOME) \( -\ln |1 - (\frac{y}{x})^2| = \ln |x| + C \) and \( y = x \) and \( y = -x \)  (b) (SEP) \( y^2 + 1 = Cx \)  
(c) (EXE) \( x^2 + 2xy^2 = C \)  (10) \( \frac{1}{2} \left( \frac{y}{x} \right)^2 = \ln |x| + C \)  
(11) \( x^2y + x + y^2 = 1 \)  
(12) \( y(2) = \frac{3}{2}e^{-2} \)  
(13) See below:

(14) (a) \( t > 0 \)  (b) \(-\frac{\pi}{2} < t < \frac{\pi}{2}\)  (c) \(-3 < x < 2\)  (d) \(-4 < x < 0\)  
(15) (a) all \( (t_0, y_0) \)  (b) all \( (t_0, y_0) \) with \( y_0 \neq 0 \)  (c) all \( (t_0, y_0) \neq (0, 0) \)  (d) all \( (t_0, y_0) \) with \( y_0 \neq 0 \)  
(e) all \( (t_0, y_0) \) where \(-1 < y_0 < 1 \) and \( t_0 \neq 2 \)  
(16) \( y = \frac{1 + 3e^{2x}}{1 - 3e^{2x}}, \) solution defined for \(-\frac{1}{2} \ln 3 < x < \infty \).  
(17) \[
\begin{align*}
\begin{cases} 
  y' = ky \\
  y(0) = 4 \\
  y(2) = 2
\end{cases} \quad \begin{cases} 
  y = 4e^{0.5t^{2}/2}, \\
  t = \frac{2\ln 0.75}{\ln 0.5} \approx 0.83
\end{cases}
\end{align*}
\]
(18) \[
\begin{align*}
\begin{cases} 
  T' = k(T - 70) \\
  T(0) = 36 \\
  T(5) = 50
\end{cases} \quad \begin{cases} 
  T = 70 - 34e^{\ln(10/17)t/5}, \\
  T(10) \approx 58.2^\circ
\end{cases}
\end{align*}
\]
(19) \[
\begin{align*}
\begin{cases} 
  Q' = 15 - \frac{3Q}{100 + 2t} \\
  Q(0) = 40
\end{cases} \quad \begin{cases} 
  Q = 3(100 + 2t) - 260,000(100 + 2t)^{-3/2}, \\
  \frac{Q(200)}{500} \approx 2.95 \text{ lbs/gal}
\end{cases}
\end{align*}
\]
(20) If $Q(t) = \#$ gals of antifreeze, then $Q' = -\frac{Q}{60}$. $Q(0) = 180$ and so $Q(t) = 180e^{-\frac{t}{60}}$. Hence $t = -60\ln\frac{5}{6} \approx 10.94$ minutes.

\[
\begin{cases}
4\frac{dv}{dt} = 128 - 8v \\
v(0) = 0
\end{cases}; \quad v = 16(1 - e^{-2t})
\]

(21) \(\frac{dy}{dt} = \frac{2y}{y_0 - (y_0 - 2)e^{2t}}\)

(22) (a) $y = 0$ and $y = 2$ (b) $y = 0$ is stable, $y = 2$ is unstable  (c) See below

(d) $y = \frac{2y_0}{y_0 - (y_0 - 2)e^{2t}}$  (e) The solution is valid for all $t$ if $0 \leq y_0 \leq 2$. If $y_0 > 2$ or $y_0 < 0$, the solution is valid only for $-\infty < t < \frac{1}{2} \ln \left( \frac{y_0}{y_0 - 2} \right)$.

(23) (a) $y = 1$ and $y = 3$ (b) only $y = 1$ is stable (c) See below

![Diagram 22(c)]

![Diagram 23(c)]