1. Compute the directional derivative of \( f(x, y, z) = x^2y + xe^z \) at \((1, -1, 0)\) in the direction from \((1, -1, 0)\) towards the origin. Is \( f \) increasing or decreasing?

2. In which unit direction(s) \( \mathbf{u} \) is the directional derivative of \( f(x, y) = (x^2 + 3y)^2 \) at \((1, 0)\) equal to zero?

3. **Page 172:** \# 15, 25.

4. **Page 176:** \# 22.

5. Show that the ellipsoid \( 6x^2 + y^2 + 6y + z^2 = 15 \) and the graph of the surface \( z = g(x, y) = x^2e^y - 4 \) are tangent at the point \((1, 0, -3)\).

6. Let \( \Gamma_1 \) and \( \Gamma_2 \) be curves which are parameterized by

\[
\mathbf{c}_1(t) = (2t^2, t + 1, t - 1) \quad \text{and} \quad \mathbf{c}_2(t) = (t + 2, 2t + 2, t^3),
\]

respectively. The point \((2, 2, 0)\) lies on both \( \Gamma_1 \) and \( \Gamma_2 \). Find a unit vector which is orthogonal to both these curves at \((2, 2, 0)\).

7. **Page 191:** \# 3, 16, 19.
1. \( \vec{v} = -(1, -1, 0) = (-1, 1, 0) \Rightarrow \hat{u} = \frac{\vec{v}}{11\sqrt{11}} = \frac{1}{\sqrt{2}} (-1, 1, 0) \)

\( f(x, y, z) = x^2 y + x e^z \Rightarrow \nabla f = (2xy + e^z, x^2, xe^z) \)

\( D_{\hat{u}} f(1, -1, 0) = \nabla f(1, -1, 0) \cdot \hat{u} = (-1, 1, 0) \cdot (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0) = \frac{2}{\sqrt{2}} \)

\( \Rightarrow f \) is increasing in this direction.

2. \( f(x, y) = (x^2 + 3y)^2 \), \( \nabla f = \left( 4x(x^2 + 3y), 6(x^2 + 3y) \right) \)

\( \hat{u} = (u_1, u_2) \)

\( O = D_{\hat{u}} f(1, 0) = \nabla f(1, 0) \cdot (u_1, u_2) = (4, 6) \cdot (u_1, u_2) = 4u_1 + 6u_2 \)

\( \Rightarrow \begin{cases} 
4u_1 + 6u_2 = 0 \\
u_1^2 + u_2^2 = 1 \quad (\text{since } 11\sqrt{11} = 1) 
\end{cases} \)

\( \Rightarrow u_2 = \pm \frac{2}{\sqrt{13}} \Rightarrow u_1 = \mp \frac{3}{\sqrt{13}} \)

\( \Rightarrow \hat{u} = \left( \frac{3}{\sqrt{13}}, -\frac{2}{\sqrt{13}} \right) \quad \text{or} \quad \hat{u} = \left( -\frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}} \right) \)
\( \vec{r} = (x, y, z) \), \( r = \|\vec{r}\| = (x^2 + y^2 + z^2)^{1/2} \)

\[ \nabla \left( \frac{1}{r} \right) = \nabla \left( (x^2 + y^2 + z^2)^{-1/2} \right) = \left( \frac{-x}{(x^2 + y^2 + z^2)^{3/2}}, \frac{-y}{(x^2 + y^2 + z^2)^{3/2}}, \frac{-z}{(x^2 + y^2 + z^2)^{3/2}} \right) \]

\[= -\frac{1}{(x^2 + y^2 + z^2)^{3/2}} \frac{(x, y, z)}{\|\vec{r}\|^3} = -\frac{\vec{r}}{r^3}. \]

\[ \text{Page 173 #25:} \quad \text{Surface } x^2 + y^2 - z^2 = -1 \]

Let \( F(x, y, z) = x^2 + y^2 - z^2 \) normal vector to surface @ \((1, 1, \sqrt{3})\)

Corresponding line particle moves along (with speed 10) is

\[ \vec{r}(t) = (1, 1, \sqrt{3}) + t \left\{ \frac{\vec{n}}{\|\vec{n}\|} \right\} = (1, 1, \sqrt{3}) + \frac{10}{\sqrt{20}} (2, 2, -2\sqrt{3}) \]

\[ \Rightarrow \begin{cases} x = 1 + \frac{20}{\sqrt{20}} t \\ y = 1 + \frac{20}{\sqrt{20}} t \\ z = \sqrt{3} - \frac{20\sqrt{3}}{\sqrt{20}} t \end{cases} \]

Particle hits \( xy \) plane when \( z = 0 \) i.e., when \( t^* = \frac{1}{\sqrt{20}} \)

Point is \( \vec{r}(t^*) = (2, 2, 0) \)
$W = x^2 + xy$

$\nabla W(-1,1) = (\frac{\partial}{\partial x} y, \frac{\partial}{\partial y} y)(-1,1) = (-1, 1)$

$W$ increases fastest in the direction $\nabla W(-1,1) = (-1, 1)$.

$\|\nabla W(-1,1)\| = \sqrt{2}$.

The directional derivative of $W$ in any direction from $(-1,1)$ is never larger than $\sqrt{2}$, and never smaller than $-\sqrt{2}$:

$-\sqrt{2} = -\|\nabla W(1,1)\| \leq D_{u} W(-1,1) \leq \|\nabla W(-1,1)\| = \sqrt{2}$
\[
\text{let } F(x, y, z) = x^3 + y^2 - xz^2 - 4x \, .
\]

Then \( \nabla \) is \( F(x, y, z) = 1 \).

(a) \[
\frac{2x}{\partial y} = - \frac{\partial F}{\partial y} = - \frac{2y}{3x^2 - z^2 - y}.
\]

(b) \[
\frac{2y}{\partial x} = - \frac{\partial F}{\partial x} = - \frac{1}{\partial y} = - \frac{(3x^2 - z^2 - y)}{2y}.
\]

(c) \[
\frac{2z}{\partial x} = - \frac{\partial F}{\partial z} = - \frac{(3x^2 - z^2 - y)}{-2xz} = \frac{3x^2 - z^2 - y}{2xz}.
\]

\[
\frac{2z}{\partial y} = - \frac{\partial F}{\partial y} = - \frac{2y}{2xz} = \frac{y}{xz}.
\]

\[
\nabla z = \left( \frac{3x^2 - z^2 - y}{2xz}, \frac{y}{xz} \right).
\]
\[ \tilde{c}_1(t) = (2t^2, t+1, t-1) \] Now \[ \tilde{c}_1(1) = (2,2,0) \Rightarrow t=1 \]

\[ \tilde{c}_1'(1) = (4,1,1) \]

\[ \tilde{c}_2(t) = (t+2, at+a, t^3) \] Here \[ \tilde{c}_2(0) = (2,2,0) \Rightarrow t=0 \]

\[ \tilde{c}_2'(0) = (1,2,0) \]

A vector \( \mathbf{u} \) to both tangent vectors is

\[ \mathbf{v} = \tilde{c}_1'(1) \times \tilde{c}_2'(0) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 1 & 1 \\ 1 & 2 & 0 \end{vmatrix} = (-2,1,7) \]

\[ \mathbf{u} = \frac{(-2,1,7)}{\sqrt{54}} \quad \text{or} \quad \mathbf{u} = \frac{(2,-1,-7)}{\sqrt{54}} \]
\[ \begin{align*} \frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} = \frac{\partial w}{\partial x} (1) + \frac{\partial w}{\partial y} (1) \\
&= \frac{\partial^2 w}{\partial v \partial u} = \frac{\partial}{\partial v} \left( \frac{\partial w}{\partial u} \right) = \frac{\partial}{\partial v} \left( \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} \right) \\
&= \frac{\partial}{\partial v} \left( \frac{\partial w}{\partial x} \right) + \frac{\partial}{\partial v} \left( \frac{\partial w}{\partial y} \right) \\
&= \left\{ \frac{\partial}{\partial x} \left( \frac{\partial w}{\partial x} \right) \frac{\partial x}{\partial v} + \frac{\partial}{\partial y} \left( \frac{\partial w}{\partial x} \right) \frac{\partial y}{\partial v} \right\} + \left\{ \frac{\partial}{\partial x} \left( \frac{\partial w}{\partial y} \right) \frac{\partial x}{\partial v} + \frac{\partial}{\partial y} \left( \frac{\partial w}{\partial y} \right) \frac{\partial y}{\partial v} \right\} \\
&= \frac{\partial^2 w}{\partial x^2} (1) + \frac{\partial^2 w}{\partial y \partial x} (-1) + \frac{\partial^2 w}{\partial x \partial y} (1) + \frac{\partial^2 w}{\partial y^2} (-1) \\
&= \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} \cdot \end{align*} \]
\[ u(x,y) = x^3 - 3xy^2 \]

\[ u_x = 3x^2 - 3y^2 \]
\[ u_{xx} = 6x \]

\[ u_y = -6xy \]
\[ u_{yy} = -6x \]

Thus, \( u_{xx} + u_{yy} = 0 \) and \( u \in C^2 \). Hence \( u \) is harmonic.