(1) **Autonomous Equations**: Equations of the form \( \frac{dy}{dt} = F(y) \) (\( \star \)) are said to be autonomous since \( \frac{dy}{dt} \) does not depend on the independent variable \( t \). Such equations can have constant solutions (i.e., \( y = K \)) which are called equilibrium solutions. These solutions are found by solving \( F(y) = 0 \). (These are also called critical points.) You should be able to find all equilibrium solutions to d.e. of the form \( \star \) and classify the stability of these equilibrium solutions as follows:

(a) **Asymptotically Stable** - Solutions which start near \( y = K \) will always approach \( y = K \) as \( t \to \infty \):

(b) **Asymptotically Unstable** - Solutions which start near \( y = K \) does not always approach \( y = K \) as \( t \to \infty \):

(c) **Semistable** - This is a special type of unstable solution. In this case solutions on one side of \( y = K \) will approach \( y = K \) as \( t \to \infty \), while solutions on the other side of \( y = K \) will approach something else:

Remark. To sketch non-equilibrium solutions of \( \star \), you do not necessarily need direction fields, you can use ordinary calculus. Since \( \frac{dy}{dt} = F(y) \), the graph of \( F(y) \) vs \( y \) will determine where the solution \( y = \phi(t) \) is increasing \( (F(y) > 0) \) or decreasing \( (F(y) < 0) \). By the Chain Rule, \( \frac{d^2y}{dt^2} = \frac{dF(y)}{dy}F(y) \), hence a graph of \( \frac{dF}{dy} \) will determine where the solution \( y = \phi(t) \) is concave up \( (F'F > 0) \) or concave down \( (F'F < 0) \).
(2) **Numerical Methods (Euler Tangent Line Method).** You should be able to compute by hand the first few approximations to the IVP \[
\begin{align*}
\frac{dy}{dt} &= f(t, y) \\
y(t_0) &= y_0
\end{align*}
\] using the Euler (Tangent Line) Method:

\[
y_n = y_{n-1} + hf(t_{n-1}, y_{n-1}) \quad t_k = t_0 + kh
\]

For the Euler Method, you can tell if the Euler approximation is smaller or larger than the true solution \(\phi(t)\) near \(t_0\) by looking at the sign of \(\frac{d^2y}{dt^2}\) at \(x_0\):

- \(\frac{d^2y}{dt^2} > 0\) at \(t_0\) \(\implies\) \(y(t)\) concave up at \(t_0\) \(\implies\) EULER approximation \(<\) \(\phi(t)\) near \(t_0\)
- \(\frac{d^2y}{dt^2} < 0\) at \(t_0\) \(\implies\) \(y(t)\) concave down at \(t_0\) \(\implies\) EULER approximation \(>\) \(\phi(t)\) near \(t_0\)

(3) **Second Order Linear Homogeneous with Equations Constant Coefficients.**

The differential equation \(ay'' + by' + cy = 0\) has Characteristic Equation \(ar^2 + br + c = 0\). Call the roots \(r_1\) and \(r_2\). The general solution to \(ay'' + by' + cy = 0\) is as follows:

- (a) If \(r_1, r_2\) are real and distinct \(\implies\) \(y = C_1 e^{r_1t} + C_2 e^{r_2t}\)
- (b) If \(r_1 = r_2\) (repeated roots) \(\implies\) \(y = C_1 e^{r_1t} + C_2 te^{r_1t}\)
- (c) If \(r_1 = \lambda + i\mu\) (hence \(r_2 = \lambda - i\mu\) \(\implies\) \(y = C_1 e^{\lambda t} \cos \mu t + C_2 e^{\lambda t} \sin \mu t\)

(4) **Theory of 2nd Linear Order Equations.** The Wronskian is defined as

\[
W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\
y_1'(t) & y_2'(t) \end{vmatrix}.
\]

- (a) The functions \(y_1(t)\) and \(y_2(t)\) are linearly independent over \(a < t < b\) if \(W(y_1, y_2) \neq 0\) for at least one point in the interval.
- (b) **THEOREM (Existence & Uniqueness)** If \(p(t), q(t)\) and \(g(t)\) are continuous in an open interval \(a < t < b\) containing \(t_0\), then the IVP

\[
\begin{align*}
y'' + p(t)y' + q(t)y &= g(t) \\
y(t_0) &= y_0 \\
y'(t_0) &= y_0'
\end{align*}
\]

has a unique solution \(y = \phi(t)\) defined in the interval \(a < t < b\).
(c) **Superposition Principle** If \(y_1(t)\) and \(y_2(t)\) are solutions to the \(n^{th}\) order linear homogeneous equation \(a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = 0\) over the interval \(a < t < b\), then \(y = k_1y_1(t) + k_2y_2(t)\) are also solutions for any constants \(k_1\) and \(k_2\).

(d) **THEOREM (Homogeneous)** If \(y_1(t)\) and \(y_2(t)\) are solutions to the homogeneous equation \(y'' + p(t)y' + q(t)y = 0\) in some interval \(I\) and \(W(y_1, y_2) \neq 0\) for some \(t\) in \(I\), then the general solution is \(y = C_1y_1(t) + C_2y_2(t)\). (We say that \(y_1(t), y_2(t)\) form a *Fundamental Set of Solutions* to the differential equation.)

(e) **THEOREM (Nonhomogeneous)** If \(y_c(t) = C_1y_1(t) + C_2y_2(t)\) is the general solution to the homogeneous equation \(y'' + p(t)y' + q(t)y = 0\) and \(y_p(t)\) is a particular solution to the nonhomogeneous equation \(y'' + p(t)y' + q(t)y = g(t)\), then the general solution to the nonhomogeneous equation is \(y = y_c + y_p = \{C_1y_1(t) + C_2y_2(t)\} + y_p(t)\).

(f) **Useful Remark**: If \(y_{p1}(t)\) is a particular solution of \(y'' + p(t)y' + q(t)y = g_1(t)\) and if \(y_{p2}(t)\) is a particular solution of \(y'' + p(t)y' + q(t)y = g_2(t)\), then

\[
y_p(t) = y_{p1}(t) + y_{p2}(t)
\]

is a particular solution of \(y'' + p(t)y' + q(t)y = g_1(t) + g_2(t)\).

(5) **Finding A Particular Solution To Nonhomogeneous Equations**. You can always use the method of Variation of Parameters to find a particular solution \(y_p(t)\) to the linear nonhomogeneous equation \(y'' + p(t)y' + q(t)y = g(t)\). However, if the coefficients are constants rather than functions AND \(g(t)\) has a very special form (see table below), it is usually easier to use Undetermined Coefficients:

(a) **Undetermined Coefficients** - **IF** \(ay'' + by' + cy = g(t)\) **AND** \(g(t)\) as below:

<table>
<thead>
<tr>
<th>(g(t))</th>
<th>Form of (y_p(t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P_n(t) = a_0t^n + a_1t^{n-1} + \cdots + a_n)</td>
<td>(t^s(A_0t^n + A_1t^{n-1} + \cdots + A_n))</td>
</tr>
<tr>
<td>(e^{\alpha t} P_n(t))</td>
<td>(t^s e^{\alpha t}(A_0t^n + A_1t^{n-1} + \cdots + A_n))</td>
</tr>
<tr>
<td>(e^{\alpha t} P_n(t) \cos \beta t) OR (e^{\alpha t} P_n(t) \sin \beta t)</td>
<td>(t^s e^{\alpha t} [F_n(t) \cos \beta t + G_n(t) \sin \beta t])</td>
</tr>
</tbody>
</table>

where \(s = \text{the smallest nonnegative integer } (s = 0, 1 \text{ or } 2)\) such that no term of \(y_p(t)\) is a solution of the corresponding homogeneous equation. In other words, no term of \(y_p(t)\) is a term of \(y_c(t)\). (\(F_n(t), G_n(t)\) are both polynomials of degree \(n\).)
Given \( y'' + p(t) y' + q(t) y = g(t) \) AND given two independent solutions \( y_1(t) \) and \( y_2(t) \) of the corresponding homogeneous equation \( y'' + p(t) y' + q(t) y = 0 \), then a particular solution \( y_p(t) \) of the nonhomogeneous equation has the form is

\[
y_p(t) = u_1(t) y_1(t) + u_2(t) y_2(t)
\]

where

\[
u_1' = \frac{0 \quad y_2}{W(y_1, y_2)} \quad \text{and} \quad u_2' = \frac{y_1 \quad 0}{W(y_1, y_2)}
\]

\[
W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}
\]

(6) **Reduction of Order.** If \( y_1(t) \) is one solution of \( y'' + p(t) y' + q(t) y = 0 \), then a second solution may be obtained using the substitution

\[
y = v(t) y_1(t)
\]

Plug this back into original differential equation to obtain a differential equation which involves \( v'(t) \) and \( v''(t) \). Reduce that equation to a 1\(^{st} \) order equation using the substitution \( w(t) = v'(t) \). Solve this 1\(^{st} \) order equation for \( w(t) \), then plug this value of \( w(t) \) back into the 1\(^{st} \) order equation \( w(t) = v'(t) \) and solve for \( v(t) \). A second solution to the original equation is then \( y_2(t) = v(t) y_1(t) \). (Note that this method of Reduction of Order also works if your original equation is nonhomogeneous and you are given a solution \( y_1(t) \) to the corresponding homogeneous equation.)

(7) **Spring-Mass Systems**

\[
\begin{align*}
m u'' + \gamma u' + k u &= F(t) \\
u(0) &= u_0, \quad u'(0) = v_0
\end{align*}
\]

\( m = \) mass of object \( \gamma = \) damping constant, \( k = \) spring constant

**Hooke’s Law:** \( F_s = k d \); \( w = m g \); \( F(t) = \) external force

A. Undamped Free Vibrations \( : m u'' + k u = 0 \) (Simple Harmonic Motion)

Solution \( u(t) = A \cos \omega_0 t + B \sin \omega_0 t = R \cos (\omega_0 t - \delta) \), where \( R = \sqrt{A^2 + B^2} \) = amplitude,

\( \omega_0 = \) frequency, \( \frac{2\pi}{\omega_0} = \) period and \( \delta = \) phase shift where \( \tan \delta = \frac{B}{A} \) (\( \delta \) depends on which quadrant the point \((A, B)\) lies).

B. Damped Free Vibrations \( : m u'' + \gamma u' + k u = 0 \)

(i) \( \gamma^2 - 4km > 0 \) (overdamped) \( \Rightarrow \) distinct real roots to CE

(ii) \( \gamma^2 - 4km = 0 \) (critically damped) \( \Rightarrow \) repeated roots to CE

(iii) \( \gamma^2 - 4km < 0 \) (underdamped) \( \Rightarrow \) complex roots to CE (motion is oscillatory)

C. Forced Vibrations \( : (F(t) = F_0 \cos \omega t \) or \( F(t) = F_0 \sin \omega t \), for example)

(i) \( m u'' + k u = F_0 \cos \omega t \) (no damping) If \( \omega = \omega_0 = \sqrt{\frac{k}{m}} \) \( \Rightarrow \) Resonance occurs and the solution is unbounded; while if \( \omega \neq \omega_0 \) then motion is a series of beats (solution is bounded)

(ii) \( m u'' + \gamma u' + k u = F(t) \) (damping) If you write the general solution as \( u(t) = u_T(t) + u_S(t) \), then \( u_T(t) = \text{Transient Solution} \); \( u_S(t) = \text{Steady-State Solution} \). (Note: transient solution satisfies \( u_T(t) \rightarrow 0 \) as \( t \rightarrow \infty \))
Practice Problems

1. Consider the differential equation $\frac{dy}{dt} = y(y - 2)$.

(a) What are the equilibrium solutions?

(b) Which equilibrium solutions are stable/unstable?

(c) Sketch the graph of the solution of the differential equation for $t \geq 0$ with each of the initial values $y(0) = -2/3$, $y(0) = 0$, $y(0) = 2/3$, $y(0) = 4/3$, $y(0) = 2$, $y(0) = 8/3$.

(d) Find the explicit solution of the initial value problem $\frac{dy}{dt} = y(y - 2)$, $y(0) = y_0$.

(e) For what values of $t$ is the solution in (d) valid?

2. Consider the differential equation $\frac{dy}{dt} = F(y)$, where the graph of $F(y)$ is indicated below.

(a) What are the equilibrium solutions?

(b) Which equilibrium solutions are stable?

(c) Sketch the graph of the solution of the differential equation with each of the initial values $y(0) = 0$, $y(0) = 1$, $y(0) = 2$, $y(0) = 3$, $y(0) = 4$.

3. Estimate the solution at $t = 1.5$ to the IVP $y' = 2t - 5y$, $y(1) = -2$ using the Euler Method with $h = 0.25$. What is the true solution at $t = 1.5$?

4. Is the Euler approximation to the IVP $\frac{dy}{dt} = y^2 - ty$, $y(-1) = 2$ smaller or greater than the true solution near $t = -1$?

5. (a) Show that $y_1 = t$ and $y_2 = t^{-1}$ are solutions of the differential equation $t^2y'' + ty' - y = 0$.

(b) Evaluate the Wronskian $W(y_2, y_1)$ at $t = \frac{1}{2}$.

(c) Find the solution of the initial value problem $t^2y'' + ty' - y = 0$, $y(1) = 2$, $y'(1) = 4$.

6. Find the largest open interval for which the initial value problem

$$3t^2y'' + y' + \frac{1}{t - 2}y = \frac{1}{t - 3}$$

$y(1) = 3$, $y'(1) = 2$, has a solution.

In Problems 7, 8, and 9 find the general solution of the homogeneous differential equations in (a) and use the method of Undetermined Coefficients to find the form of a particular solution of the nonhomogeneous equations in (b) and (c).

7. (a) $y'' - 5y' + 6y = 0$ (b) $y'' - 5y' + 6y = t^2$ (c) $y'' - 5y' + 6y = e^{2t} + \cos(3t)$

8. (a) $y'' - 6y' + 9y = 0$ (b) $y'' - 6y' + 9y = te^{3t}$ (c) $y'' - 6y' + 9y = e^t + \cos(3t)$

9. (a) $y'' - 2y' + 10y = 0$ (b) $y'' - 2y' + 10y = e^x + \cos(3x)$ (c) $y'' - 2y' + 10y = e^x \cos(3x)$
10. Find the general solution of the differential equation $y'' - y' = 4t$.
11. Find the general solution to $y'' + y = \tan t$, $0 < t < \frac{\pi}{2}$.
12. The differential equation $t^2 y'' + ty' - y = 0$ has one solution $y_1(t) = t$. Use the method of **Reduction of Order** to find a second (linearly independent) solution of $t^2 y'' + ty' - y = 0$.
13. Solve $y'' + 4y' = -10 \cos 2t$, $y(0) = 0$, $y'(0) = 2$.
14. Find a fundamental set of solutions of $4y'' + 12y' + 9y = 0$.
15. For what nonnegative values of $\gamma$ will the solution of the initial value problem $u'' + \gamma u' + 4u = 0$, $u(0) = 4$, $u'(0) = 0$ oscillate?
16. (a) For what positive values of $k$ does the solution of the initial value problem $2u'' + ku = 3 \cos 2t$, $u(0) = 0$, $u'(0) = 0$, become **unbounded** (Resonance)?
(b) For what positive values of $k$ does the solution of the initial value problem $2u'' + u' + ku = 3 \cos 2t$, $u(0) = 0$, $u'(0) = 0$, become **unbounded** (Resonance)?
17. Find the steady-state solution of the IVP $y'' + 4y' + 4y = \sin t$, $y(0) = 0$, $y'(0) = 0$.
18. A 4-kg mass stretches a spring 0.392 m. If the mass is released from 1 m below the equilibrium position with a downward velocity of 10 m/sec, what is the maximum displacement?

**Answers**

(1) (a) $y = 0$ and $y = 2$  (b) $y = 0$ is stable, $y = 2$ is unstable  (c) **See below**
(d) $y = \frac{2y_0}{y_0 - (y_0 - 2)e^{2t}}$  (e) The solution is valid for all $t$ if $0 \leq y_0 \leq 2$. If $y_0 > 2$ or $y_0 < 0$, the solution is valid only for $-\infty < t < \frac{1}{2} \ln \left( \frac{y_0}{y_0 - 2} \right)$.
(2) (a) $y = 1$ and $y = 3$  (b) only $y = 1$ is stable  (c) **See below**
(3) $y_2 = 0.375$, true solution $\phi(1.5) = \frac{1}{25}(13 - 58 e^{-2.5}) \approx 0.3296$
(4) $y''(-1) > 0$ thus **EULER** approximation, near $t = -1$
(5) (b) $W(t^{-1}, t)(\frac{1}{2}) = 4$  (c) $y = 3t - t^{-1}$  (6) $0 < t < 2$
(7) (a) \( y = C_1 e^{2t} + C_2 e^{3t} \) (b) \( y = At^2 + Bt + C \) (c) \( y = Ate^{2t} + B \cos(3t) + C \sin(3t) \)
(8) (a) \( y = C_1 e^{3t} + C_2 te^{3t} \) (b) \( y = t^2(At + B)e^{3t} \) (c) \( y = Ae^t + B \cos(3t) + C \sin(3t) \)
(9) (a) \( y = C_1 e^x \cos(3x) + C_2 e^x \sin(3x) \) (b) \( y = Ae^x + B \cos(3x) + C \sin(3x) \) (c) \( y = x(A \cos(3x) + B \sin(3x))e^x \)
(10) \( y = C_1 + C_2 e^t - 2t^2 - 4t \) (11) \( y = C_1 \cos t + C_2 \sin t - (\cos t) \ln(\sec t + \tan t) \)
(12) \( y = t^{-1} \) or \( y = At^{-1} + Bt, A \neq 0 \) (13) \( y = \frac{1}{2} - e^{-4t} + \left( \frac{1}{2} \cos 2t - \sin 2t \right) \)
(14) \( \{ e^{-\frac{2}{t}}, te^{-\frac{2}{t}} \} \) (15) \( 0 \leq \gamma < 4 \)
(16) (a) \( k = 8 \) (resonance) (b) NO value of \( k \), all solutions are bounded.
(17) \( y = \frac{1}{25}(3 \sin t - 4 \cos t) \)
(18) \( u(t) = \cos 5t + 2 \sin 5t = \sqrt{5} \cos(5t - \delta), \ \delta = \tan^{-1}2 \approx 1.1 \) Thus amplitude = \( \sqrt{5} \).