(1) **n**\(^{th}\) Order Linear Homogeneous Equations With Constant Coefficients

\[a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0 \quad (\ast)\]

**Characteristic Equation:** \[a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0\] factor into roots that are real and distinct, repeated, complex, or complex and repeated.

(a) If \(r_1\) = real and not repeated \(\Rightarrow\) get one solution of (\(\ast\)) : \(e^{r_1 t}\)

(b) If \(r_1\) = real and repeated \(m\) times \(\Rightarrow\) get \(m\) independent solutions of (\(\ast\)) : 

\[e^{r_1 t}, te^{r_1 t}, \ldots, t^{m-1}e^{r_1 t}\]

(c) If \(r_1 = \lambda + i\mu\) = complex and repeated \(m\) times \(\Rightarrow\) get \(2m\) solutions of (\(\ast\)):

\[e^{\lambda t} \cos \mu t, te^{\lambda t} \cos \mu t, \ldots, t^{m-1}e^{\lambda t} \cos \mu t \quad \text{and} \quad e^{\lambda t} \sin \mu t, te^{\lambda t} \sin \mu t, \ldots, t^{m-1}e^{\lambda t} \sin \mu t\]

(2) **Undetermined Coefficients for n**\(^{th}\) **Order Linear Equations**

This can only be used to find a particular solution \(y_p(t)\) when

\[a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = g(t)\]

and \(g(t)\) one of the 3 very SPECIAL FORMS as in the case of 2\(^{nd}\) order equations. The particular solution has the same form as before : \(y_p(t) = t^s [ \cdots ]\), except this time \(s = 0, 1, 2, \ldots, n\), where \(s\) = the smallest integer such that NO term of \(y_p(t)\) is a term of the general solution \(y_c(t)\) of the corresponding homogeneous equation.

(3) **Laplace Transforms:**

A. Be able to compute Laplace transforms using definition:

\[\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) \, dt\]

and using a table of Laplace transforms (see table on page 304) and using linearity:

\[\mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}, \quad \mathcal{L}\{c f(t)\} = c \mathcal{L}\{f(t)\}.\]

B. Computing Inverse Laplace Transforms: Must be able to use a table of Laplace transforms usually together with Partial Fractions or Completing the Square, to find inverse Laplace transforms: \(f(t) = \mathcal{L}^{-1}\{F(s)\}\).

C. Solving Initial Value Problems: Just need to recall that

\[\mathcal{L}\{y\}' = s \mathcal{L}\{y\} - y(0)\]

\[\mathcal{L}\{y''\} = s^2 \mathcal{L}\{y\} - s y(0) - y'(0)\]

\[\mathcal{L}\{y'''\} = s^3 \mathcal{L}\{y\} - s^2 y(0) - s y'(0) - y''(0)\]

\[\vdots\]
D. Discontinuous Functions:

(i) **Unit Step Functions**: For \( c \geq 0 \), \( u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases} \)

\[
\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}
\]

![Diagram of Unit Step Function](image)

(ii) **Translated functions**: \( y = g(t) = \begin{cases} 0, & t < c \\ f(t - c), & t \geq c \end{cases} = u_c(t) f(t - c) \).

![Diagram of Translated Functions](image)

\[
\mathcal{L}\{u_c(t) f(t - c)\} = e^{-cs} \mathcal{L}\{f(t)\} = e^{-cs} F(s)
\]

Thus,

\[
\mathcal{L}^{-1}\{e^{-cs} F(s)\} = u_c(t) f(t - c), \quad \text{where } f(t) = \mathcal{L}^{-1}\{F(s)\}
\]

(iii) **Unit Impulse Functions**: If \( y = \delta(t - t_0) \ (t_0 \geq 0) \), then

\[
\mathcal{L}\{\delta(t - t_0)\} = e^{-st_0}
\]

E. **Convolutions**:

\[
\mathcal{L}\{(f \ast g)(t)\} = \mathcal{L}\left\{ \int_0^t f(t - \tau) g(\tau) \, d\tau \right\} = \mathcal{L}\{f(t)\} \mathcal{L}\{g(t)\}
\]
In Problems 1 and 2 find the general solution of the homogeneous differential equations in (a) and use the method of Undetermined Coefficients to find the form of a particular solution of the nonhomogeneous equation in (b).

1. (a) $y'' - y' = 0$  
   (b) $y'' - y' = t + e^t$

2. (a) $y'' - y'' - y' + y = 0$  
   (b) $y'' - y'' - y' + y = e^t + \cos t$

3. Find the solution of the initial value problem $y'' - 2y' + y = 0$, $y(0) = 2$, $y'(0) = 0$, $y''(0) = 1$.

4. Find the general solution of the differential equation $y'' + y' = t^2$.

5. Find a fundamental set of solutions of $y^{(5)} - 4y'' = 0$.

6. Find the Laplace transform of these functions:
   (a) $f(t) = 3 - e^{2t}$  
   (b) $g(t) = 100t^5$  
   (c) $h(t) = \cosh \pi t$  
   (d) $k(t) = -10t^3 e^{5t}$

7. Find the inverse Laplace transform of these functions:
   (a) $F(s) = \frac{9}{s^2 - s - 2}$  
   (b) $F(s) = \frac{s}{(s - 1)^2}$  
   (c) $F(s) = \frac{8}{(s + 1)^4}$

8. Solve these initial value problems:
   (a) $y'' - y' - 6y = 0$  
   $y(0) = 1$  
   $y'(0) = -1$

   (b) $y'' - 2y' + 2y = \cos t$  
   $y(0) = 1$  
   $y'(0) = 0$

   (c) $y'' - y = \begin{cases} 1, & t < 5 \\ 2, & 5 \leq t < \infty \end{cases}$  
   $y(0) = y'(0) = 0$.

   (d) $y'' + 4y = \begin{cases} t, & t < 1 \\ 0, & 1 \leq t < \infty \end{cases}$  
   $y(0) = y'(0) = 0$.

   (e) $y' + y = g(t)$, $y(0) = 0$ and where $g(t)$:

   (f) $y'' + 4y = \delta(t - 3)$, $y(0) = y'(0) = 0$

9. $\mathcal{L}\left\{ \int_0^t 100 e^{-2\tau} \cos \pi(t - \tau) d\tau \right\} = ?$

10. If $g(t) = \mathcal{L}^{-1}\{G(s)\}$, then $\mathcal{L}^{-1}\left\{ \frac{G(s)}{(s - 3)^2} \right\} = ?$
Answers

(1) (a) \(y = C_1 + C_2 e^{-t} + C_3 e^t\) \hspace{1cm} (b) \(y = t(At + B) + Cte^t\)

(2) (a) \(y = C_1 e^t + C_2 te^t + C_3 e^{-t}\) \hspace{1cm} (b) \(y = At^2 e^t + B \cos t + C \sin t\)

(3) \(y = 3 - e^t + te^t\)

(4) \(y = C_1 + C_2 \cos t + C_3 \sin t + \frac{1}{3}t^3 - 2t\)

(5) \(\{1, t, t^2, e^{2t}, e^{-2t}\}\)

(6) (a) \(\frac{2s - 6}{s^2 - 2s}\) \hspace{1cm} (b) \(\frac{12000}{s^6}\) \hspace{1cm} (c) \(\frac{s}{s^2 - \pi^2}\) \hspace{1cm} (d) \(-\frac{60}{(s - 5)^4}\)

(7) (a) \(3(e^{2t} - e^{-t})\) \hspace{1cm} (b) \(e^t + te^t\) \hspace{1cm} (c) \(\frac{4}{3}t^3 e^{-t}\)

(8) (a) \(y = \frac{1}{5}(e^{3t} + 4e^{-2t})\) \hspace{1cm} (b) \(y = \frac{1}{5}(\cos t - 2 \sin t + 4e^t \cos t - 2e^t \sin t)\)

(c) \(y = -1 + \frac{1}{2}(e^t + e^{-t}) + u_5(t)(-1 + \frac{1}{2}(e^{(t-5)} + e^{-(t-5)}))\),

or \(y = -1 + \cos t + u_5(t)(-1 + \cosh(t - 5))\)

(d) \(y = (-\frac{1}{8} \sin 2t + \frac{7}{4}) - u_1(t)(-\frac{1}{8} \sin 2(t - 1) + \frac{21}{4}) - u_1(t)(\frac{1}{4} - \frac{1}{4} \cos 2(t - 1))\)

(e) \(y = 3(1 - e^{-t}) - 3u_2(t)(1 - e^{-(t-2)}) + 3u_4(t)(1 - e^{-(t-4)})\)

(f) \(y = \frac{1}{2}u_3(t) \sin(2t - 3)\)

(9) \(\frac{100s}{(s + 2)(s^2 + \pi^2)}\)

(10) \(\int_0^t (t - \tau) e^{3(t-\tau)} g(\tau) d\tau\) or \(\int_0^t \tau e^{3\tau} g(t - \tau) d\tau\)