11) Let \( T \) be the linear operator on \( F^2 \) which is represented in the standard ordered basis by the matrix \[
\begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix}
\]. Let \( a_1 = (0, 1) \). Show that \( F^2 \neq Z(a_1, T) \) and that there is no non-zero vector \( a_2 \) in \( F^2 \) such that \( Z(a_2, T) \) is disjoint from \( Z(a_1, T) \).

11) Let \( T \) be a linear operator on \( V \) an \( n \)-dimensional vectorspace and let \( R = T(V) \) be the range of \( T \).
   (a) Prove that \( R \) has a complementary \( T \)-invariant subspace iff \( R \) is independent of \( N = \text{null} T \).
   (b) If \( R \) and \( N \) are independent, prove that \( N \) is the unique \( T \)-invariant subspace complementary to \( R \).

11) Let \( T \) be the linear operator on \( \mathbb{R}^3 \) which is represented by the matrix \[
\begin{pmatrix}
2 & 0 & 0 \\
1 & 2 & 0 \\
0 & 0 & 3
\end{pmatrix}
\]. Let \( W \) be the null space of \( T - 2I \). Prove that \( W \) has no complementary \( T \)-invariant subspace.

11) Let \( T \) be the linear operator on \( F^4 \) which is represented by the matrix \[
\begin{pmatrix}
c & 0 & 0 & 0 \\
0 & c & 0 & 0 \\
0 & 1 & c & 0 \\
0 & 0 & 1 & c
\end{pmatrix}
\] and let \( W \) be the nullspace of \( T - cI \).
   (a) Prove that \( W \) is the subspace spanned by \( e_4 = (1, 0, 0, 0) \).
   (b) Find the monic generators of the ideals \( S(e_4; W), S(e_3; W), S(e_2; W), S(e_1; W) \) where \( S(v, W) \) is the \( T \)-conductor of \( v \) into \( W \), i.e. the ideal of polynomials \( g(x) \) such that \( g(T)v \in W \).

11) Let \( T \) be a linear operator over a subfield of \( \mathbb{C} \) with matrix representation \[
\begin{pmatrix}
2 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 \\
0 & a & 2 & 0 \\
0 & 0 & b & 2
\end{pmatrix}
\]. Find the characteristic polynomial of \( T \). Find the minimal polynomial of \( T \) and vectors satisfying theorem 3 in each of these cases: \( a = 1 = b \); \( a = 0 = b \); \( a = 0, b = 1 \).

11) For \( A, B \in F^{3 \times 3} \), show that \( A B \) iff the characteristic and minimal polynomials of \( A \) are the same as those of \( B \).

11) Let \( F \) be a subfield of \( \mathbb{C} \) and let \( A \) and \( B \) be \( n \times n \) matrices over \( F \). Prove that if \( A \) and \( B \) are similar over \( \mathbb{C} \) then they are similar over \( F \).

11) Let \( A \) be an \( n \times n \) matrix over \( \mathbb{C} \). Prove that if every characteristic value of \( A \) is real then \( A \) is similar to a matrix with real entries.

11) Let \( T \) be a linear operator on the finite-dimensional vectorspace \( V \). Prove that there exists a vector \( v \) in \( V \) such that if \( f \) is a polynomial and \( f(T)v = 0 \) then \( f(T) = 0 \) (this is called a separating vector). When \( T \) has a cyclic vector give a direct proof that any cyclic vector is a separating vector for \( T \).

11) Let \( F \) be a subfield of \( \mathbb{C} \) and let \( A \) be an \( n \times n \) matrix of \( F \). Let \( p \) be the minimal polynomial for \( A \). If we regard \( A \) as a matrix over \( \mathbb{C} \), then \( A \) has a minimal polynomial \( f \) as an \( n \times n \) matrix over \( \mathbb{C} \). Show that \( p = f \) (using “a theorem on linear equations”). Can you prove it using the cyclic decomposition theorem?

11) Let \( T \) be a linear operator on an \( n \)-dimensional vectorspace \( V \) over \( F \). Show that if the minimal polynomial for \( T \) is a power of an irreducible polynomial and the minimal polynomial is equal to the characteristic polynomial then no non-trivial \( T \)-invariant subspace has a complementary \( T \)-invariant subspace.

11) Show that if \( T \) is a diagonalizable linear operator on \( V \) then every \( T \)-invariant subspace of \( V \) has a complementary \( T \)-invariant subspace.

11) Let \( T \) be a linear operator on the \( n \)-dimensional vectorspace \( V \). Prove that \( T \) has a cyclic vector iff every linear operator \( U \) which commutes with \( T \) is a polynomial in \( T \).

11) Let \( V \) be an \( n \)-dimensional vectorspace over the field \( F \) and let \( T \) be a linear operator on \( V \). Prove that every nonzero vector in \( V \) is a cyclic vector for \( T \) iff the characteristic polynomial for \( T \) is irreducible over \( F \).
Let $A$ be an $n \times n$ matrix over $\mathbb{R}$. Let $T$ be the linear operator on $\mathbb{R}^n$ represented by $A$, and let $U$ be the linear operator on $\mathbb{C}^n$ which is represented by $A$. Use 20 to prove that if the only subspaces invariant under $T$ are $\mathbb{R}^n$ and 0 then $U$ is diagonalizable.