Let \( V = \mathbb{C}^2 \) with the standard inner product. Let \( T \) be the linear operator defined by \( T\epsilon_1 = (1, -2), T\epsilon_2 = (i, -1) \). Let \( \alpha = (x_1, x_2) \) and find \( T^*\alpha \).

Let \( T \) be the linear operator on \( \mathbb{C}^2 \) defined by \( T\epsilon_1 = (1 + i, 2) \) and \( T\epsilon_2 = (i, i) \). Find the matrix of \( T^* \) in the standard ordered basis. Does \( T \) commute with \( T^* \)?

Show that the range of \( T^* \) is the orthogonal complement of null \( T \), i.e. show \( R = R(T^*) = (\text{null}(T))^\perp = N \).

Let \( V \) be a finite dimensional inner product space (fin dim IPS), and \( T \) a linear operator on \( V \). If \( T \) is invertible, show that \( T^* \) is invertible and that \( (T^*)^{-1} = (T^{-1})^* \).

Show that the product of 2 self-adjoint operators is self-adjoint iff the two operators commute.

Let \( V \) be a fin dim IPS over \( \mathbb{C} \). Let \( E \) be a projection operator / an idempotent operator on \( V \). Prove \( E \) is self-adjoint iff \( E \) is normal, i.e. \( E = E^* \) iff \( E^*E = EE^* \).

Let \( V \) be a fin dim IPS over \( \mathbb{C} \). Let \( T \) be a linear operator on \( V \). Show that \( T \) is self-adjoint iff \( (Tx|x) \) is real for all \( x \in V \).

For each matrix \( A \), find a real orthogonal matrix \( P \) such that \( P^TAP \) is diagonal.

Is a complex symmetric matrix self-adjoint? Is it normal?

Give an example of a \( 2 \times 2 \) matrix \( A \) such that \( A^2 \) is normal but \( A \) is not normal.

Prove that a real symmetric matrix has a real symmetric cube root.

Prove that a normal and nilpotent operator is the zero operator.

If \( T \) is a normal operator, prove that characteristic vectors for \( T \) which are associated with distinct characteristic values are orthogonal.

Let \( T \) be a normal operator on \( V \) a fin dim IPS over \( \mathbb{C} \). Prove that there is a polynomial \( f \in \mathbb{C}[x] \) such that \( T^* = f(T) \).

If two normal operators commute, prove that their product is normal.