3) Let $V$ be an $n$ dimensional vectorspace. Let $T : V \to V$ be a linear transformation such that $T(V) = \text{null } T$. Show $n$ is even.

3) Let $V$ be a vectorspace and $T : V \to V$ a linear transformation. Show that “$T(V) \cap \text{null } T = \{0\}$” $\iff$ “$T(T\alpha) = 0 \Rightarrow T\alpha = 0$”.

3) Let $V$ be a vectorspace and $T$ a linear operator on $V$. If $T^2 = 0$ but $T \neq 0$ what can we say about the relationship of the range of $T$ to the nullspace of $T$?

3) Let $V$ be a vectorspace with dim $V = n$ and let $T$ be a linear operator on $V$ such that rank $T = \text{rank } T^2$. Show that $T(V) \cap \text{null } T = \{0\}$.

3) Let $T \in L(F^n, F^n)$, let $A$ be the matrix of $T$ in the standard ordered basis for $F^n$, and let $W$ be the subspace spanned by the column vectors of $A$.

3) Find a basis for the range and the null space of $T$.

(a) The matrix for $T$ in the standard ordered basis for $\mathbb{R}^2$ is given by

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}.$$ 

(b) Using the new ordered basis $B' = \{(1, 2); (1, -1)\}$ we have

$$T' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$ 

(c) Prove that for every real number $c$ the linear operator $(T - cI)$ is invertible.

$$T - cI = T(x, y) - c(x, y) = (x, y) - (cx, cy) = (-cx - y, x - cy).$$

To compute $[(T - cI)]_B$, $(T - cI)e_1 = (-c, 1) = (-c)e_1 + 1e_2; (T - cI)e_2 = (-1, -c) = (-1)e_1 + (-c)e_2$, i.e.$[T - cI]_B = \begin{bmatrix} -c & 1 & -c \\ -1 & 0 & -c \end{bmatrix}$. But the matrix

$$M = \frac{1}{c^2 + 1} \begin{bmatrix} -c & 1 & -c \\ 1 & -c & c \end{bmatrix}$$

can be seen to be the inverse of $[(T - cI)]_B$ by multiplying them out, so the linear operator corresponding to $M$ must be the inverse of $I$. We know there exists a linear operator corresponding the the matrix $M$ by theorem 11 from chapter 3. Since $M$ is a left and right inverse to $[(T - cI)]_B$ we see that the corresponding linear operator is also the left and right inverse of $T - cI$, and so $T - cI$ is an invertible linear operator.

$d)$ Let $B' = b_1, b_2$ be any ordered basis for $\mathbb{R}^2$ and let $[T]_{B'} = A$. Show $A_{12}A_{21} \neq 0$.

Suppose by way of contradiction that $A_{12}A_{21} = 0$. By theorem 14 from ch 3 we know that there exists some invertible matrix $P$ such that $[T]_B = P^{-1}AP$ so we have $

\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} = P^{-1} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} P. \quad (1)$

We showed in part c that $T - cI = [T]_B - [cI]_B = [T]_B - cI$ is invertible for any $c$. So subtracting $cI$ from both sides of the above equation gives $[T]_B - cI = P^{-1}AP - cI$ where the left side is invertible. Multiplying both sides on the left by $P$ and the right by $P^{-1}$ gives $P[T]_B - cI)P^{-1} = A - cIP^{-1} = A - c(I$). Since each of $P, P^{-1}, [T]_B - cI$ is invertible we know their product is invertible. Hence, the right side must be invertible, for any choice of $c$. But if we let $c = A_21$, then $A - cI = \begin{bmatrix} A_{11} - A_{22} & A_{12} \\ A_{21} & 0 \end{bmatrix}$. If $A_{12}A_{21} \neq 0$ then at least one of them is equal to 0, and so either the bottom row or the right column of the matrix $A - cI$ are all zeros, and hence the matrix cannot be invertible. This is a contradiction, and hence $A_{12}A_{21} \neq 0$.

10) Let $S$ be a linear operator on $\mathbb{R}^2$ satisfying $S^2 = S$. Show that either $S = I$, $S = 0$, or an ordered basis $B$ for $\mathbb{R}^2$ exists for which the matrix representation $M$ of $S$ relative to $B$ is given by $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, i.e. $[S]_B = A$.

Suppose $S$ is not invertible. Then $S^{-1}S^2 = S^{-1}S$ because $S^2 = S$, which gives $S = I$. Now assume that $S$ is not invertible. Since $S$ is not invertible, its matrix representation $M$ (relative to any basis) is not invertible, and therefore has rank $\leq 1$. If rank $A = 0$ then $A$ is the 0 matrix and so $S$ would be the zero linear operator.

**BETTER SOLUTION:** If $S$ is not the zero linear operator, it has rank 1. Thus, $S(V)$ has rank 1 and $(S)$ has rank 1, so take the basis elements from each of those dimension 1 spaces. Together they form a basis for $\mathbb{R}^2$ because neither one is a multiple of the other, or else they would belong to each other’s spans (contradicting the range and the nullspace being distinct). Hence, $S(n_1) = 0n_1 + 0b_1$ and $S(b_1) = (b_1)b_1 + 0n_1$.

Thus, we can assume that $S$ is not the identity or zero linear operator (and so $M$ is not the 0 or identity matrix). Hence, $M$ must have rank $M > 0$, i.e. rank $M = 1$. This means that there exists a linear dependency among the columns of $M$ (since the row rank, column rank, and rank of the matrix are all the same), i.e. $k_1C_1 + k_2C_2 = 0$ for
some nonzero scalars $k_1, k_2$ and the column vectors of $A$, $C_1$ and $C_2$. But this dependency implies $C_1 = k_1^{-1}k_2C_2$ (since the scalars are nonzero we can divide), i.e. one column is a multiple of the other column. A similar proof shows that one row is a multiple of the other. So we can write $M$ (relative to the standard basis) as $M = \begin{bmatrix} a & ak \\ b & bk \end{bmatrix}$.

Now since $M^2 = M$, we know from multiplying the matrices out and comparing the first entry in each that we must have $a = a^2 + abk = a(a + bk)$. If we assume $a$ is nonzero then dividing gives us $1 = a + bk$, i.e. $1 - a = bk$. We'll use this fact in a little bit, and we'll return to the case $a = 0$ later.

For now, consider the basis $B = \{\beta_1 = (ka, 1 - a), \beta_2 = (-k, 1)\}$. Row reducing the matrix $\begin{bmatrix} -k & a \\ ak & 1 - a \end{bmatrix}$ gives $I$, so we know that these vectors are a basis. Now, note that

$$\begin{bmatrix} a & ak \\ b & bk \end{bmatrix} (ka, 1 - a) = (ka, kb)$$

but we showed above that $kb = 1 - a$, so we have $M\beta_1 = 1\beta_1 + 0\beta_2$. Similarly,

$$\begin{bmatrix} a & ak \\ b & bk \end{bmatrix} (-k, 1) = (0, 0), \text{ i.e. } M\beta_2 = 0\beta_1 + 0\beta_2.$$

This shows that the matrix representing $S$ relative to the basis $B$ is $[S]_B \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = A$ as desired.

Now we return to the case $a = 0$. Looking again at the computation $M^2 = M = \begin{bmatrix} a & ak \\ b & bk \end{bmatrix}$ we see that $b = ab + b^2k = b(a + bk)$. If $b$ is nonzero then by dividing we again have $1 - a = bk$. The proof above can proceed from here. If $b$ is also zero, then we have $a = 0$ and $b = 0$. Since we've assumed that $M$ is not the 0 matrix, one of the entries in the right column is nonzero, and we should have labelled the matrix $\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}$. With at least one of $a$ or $b$ nonzero. Again computing $M^2 = M$ gives us the equations $ab = a$ and $b^2 = b$. Hence $b$ must be 0 or 1. If $b$ is 0 then the first equation implies $a$ is zero, giving the 0 matrix, a contradiction. So $b = 1$, and $a$ can be anything. If $a$ is 0 then $B = \{\beta_1 = (0, 1), \beta_2 = (1, 0)\}$ gives $[S]_B = A$ as above and we're done.

If, on the other hand, $a$ is nonzero, then $\beta_1 = (a, 1), \beta_2 = (1, 0)$ gives $[S]_B = A$ as above and we're done. This completes all cases.

12 For $V$ a vectorspace with ordered basis $B = \{a_1, \ldots, a_n\}$ define a linear operator (via theorem 1) such that $Ta_1 = a_2, \ldots, Ta_i = a_{i+1}, \ldots, Ta_{n-1} = a_n$ and $Ta_n = 0$.

(a) Find the matrix $A$ corresponding to $T$ in basis $B$.

(b) Prove that $T^n = 0$ but $T^{n-1} \neq 0$.

(c) Let $S$ be a linear operator on $V$ satisfying $S^n = 0$ but $S^{n-1} \neq 0$. Prove there exists an ordered basis $B'$ for $V$ such that $[S]_{B'} = A$ the matrix described in part (a).

(d) Let $M$ and $N$ be any $n \times n$ matrices over $F$ such that $M^n = N^n = 0$ but $M^{n-1} \neq 0 \neq N^{n-1}$. Show $M$ and $N$ are similar.