5 Let \( T \) be the linear operator on \( \mathbb{R}^3 \) defined by \( T(x, y, z) = (x, z, -2y - z) \). Let \( f \) be the polynomial over \( \mathbb{R} \) defined by \( f = -x^3 + 2 \). Find \( f(T) \).

5 Let \( A \) be an \( n \times n \) diagonal matrix over the field \( F \). Let \( f \) be the polynomial over \( F \) defined by \( f = (x - A_{11}) \cdots (x - A_{nn}) \). What is the matrix \( f(A) \)?

5 For \( a, b \in F \) a field and \( a \neq 0 \) show that \( B = \{1, ax + b, (ax + b)^2, (ax + b)^3, \cdots \} \) is a basis for \( F[X] \).

5 If \( F \) is a field and \( h \in F[X] \) of degree \( \geq 1 \) show that the mapping \( f \mapsto f(h) \) is a one-to-one linear transformation of \( F[X] \) into \( F[X] \). Show that this transformation is an isomorphism iff \( \deg h = 1 \).

5 Use Lagrange Interpolation to find \( f \) such that \( \deg f \leq 3 \) satisfying
\[
\begin{align*}
f(-1) &= -6; f(0) = 2; f(1) = -2; f(2) = 6.
\end{align*}
\]

5 Let \( L \) be a linear functional on \( F[X] \) such that \( L(fg) = L(f)L(g) \) for all \( f, g \in F[X] \). Show that either \( L = 0 \) or there is a \( t \) in \( F \) such that \( L(f) = f(t) \) for all \( f \) in \( F[X] \).

5 If \( A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 0 \end{bmatrix} \), find the monic polynomial that generates the ideal of all polynomials \( f \in F[X] \) such that \( f(A) = 0 \).

5 Assuming the fundamental theorem of algebra, prove that if \( f \) and \( g \) are polynomials over \( \mathbb{C} \), then \( \gcd(f, g) = 1 \) iff \( f \) and \( g \) have no common root.

5 Let \( D \) be the differentiation operator. Assuming the fundamental theorem of algebra, show that for a polynomial \( f \) over \( \mathbb{C} \) we have \( f \) has no repeated roots iff \( \gcd(f, Df) = 1 \).

5 In which of these cases is \( D \) a 3-linear function?
   (a) \( D(A) = A_{11} + A_{22} + A_{33} \) 2nd row are unchanged, but \( sD(A) + D(A') = sA_{11} + sA_{22} + sA_{33} + A_{11}' + A_{22}' + A_{33}' \), which is clearly not the same for all matrices \( A \).
   (b) \( D(A) = (A_{11})^2 + 3A_{11}A_{22} \)
   (c) \( D(A) = A_{11}A_{12}A_{33} \)
   (d) \( D(A) = A_{13}A_{22}A_{32} + 5A_{12}A_{22}A_{32} \)
   (e) \( D(A) = 0 \)
   (f) \( D(A) = 1 \)

5 Let \( K \) be a subfield of \( \mathbb{C} \) and \( n \in \mathbb{N} \). Let \( j_1, \cdots, j_n \) and \( k_1, \cdots, k_n \) be positive integers not exceeding \( n \) (i.e. they can each represent a row number of an \( n \times n \) matrix over \( K \)). Let \( A \in K^{n \times n} \) and defined \( D(A) = A(j_1, k_1) \cdots A(j_n, k_n) \). Prove that \( D \) is \( n \)-linear iff the integers \( j_1, \cdots, j_n \) are distinct.

5 Let \( F \) be a field and \( D \) be a function on \( F^{n \times n} \). Suppose \( D(AB) = D(A)D(B) \) for all \( A, B \in F^{n \times n} \). Show that either \( D(A) = 0 \) for all \( A \), or \( D(I) = 1 \).