1.1 Let $F$ be a field, let $n \in \mathbb{N}$, and let $T : V \rightarrow V$ be a linear operator.
(a) Define characteristic polynomial of $T$.
(b) Define minimal polynomial of $T$.
(c) Prove that the characteristic polynomial and minimal polynomial have precisely the same roots.

1.2 Let $V$ be a vector space over an infinite field $F$. Prove that $V$ is not the union of finitely many proper subspaces.

1.3 Let $V$ be a finite-dimensional vector space over an infinite field $F$ and let $\alpha_1, \cdots, \alpha_m$ be finitely many nonzero vectors in $V$. Prove that there exists a linear functional $f$ on $V$ such that $f(\alpha_i) = 0$ for each $i$ with $1 \leq i \leq m$.

1.4 Let $V = \mathbb{R}^{4 \times 4}$. Let $D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ and define $T_D : V \rightarrow V$ by $T_D(A) = DA - AD$. List all the characteristic values of $T_D$ and the dimension of the associated characteristic space.

1.5 Let $F$ be a field and $L$ be a linear functional on the polynomial ring $F[x]$ having the property that $L(fg) = L(f)L(g)$ for all $f, g \in F[x]$. Prove that either $L = 0$ or there exists $t \in F$ such that $L(f) = f(t)$ for all $f \in F[x]$.

1.6 Let $T$ be a linear operator on $\mathbb{R}^3$ the matrix of which in the standard ordered basis is $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$ Find a basis for the range of $T$ and a basis for the null space of $T$.

1.7 Let $F$ be a field and $A, B \in F^{n \times n}$.
(a) Prove or disprove that $AB$ and $BA$ have the same characteristic polynomial.
(b) Prove or disprove that $AB$ and $BA$ have the same minimal polynomial.

1.8 Let $V$ be a 5-dimensional vector space over the field $F$ and let $T : V \rightarrow V$ be a linear operator. Assume that $c \in F$ is such that there exists a nonzero vector $\alpha$ with $T\alpha = c\alpha$. Prove that there exists a nonzero linear functional $f$ on $V$ such that $T^f f = cf$.

1.9 Let $F$ be a field and let $V = F^{5 \times 5}$. Let $W$ be the subspace of $V$ spanned by the matrices of the form $C = AB - BA$ where $A, B \in V$. Prove that $W$ is the subspace of $V$ of matrices having trace zero.

1.10 Let $A \in \mathbb{R}^{3 \times 3}$ be such that $\det A = 3$ and let $\text{adj}(A) \in \mathbb{R}^{3 \times 3}$ denote the classical adjoint of $A$.
(a) What is the product $\text{adj}(A)A$?
(b) What is $\det(\text{adj}(A))$?
(c) What is $\text{adj}(\text{adj}(A))$?