2.1 Let $p$ be a prime integer and let $F = \mathbb{Z}/(p)$ be the field with $p$ elements. Let $V$ be the $F[x]$-module $V = \frac{F[x]}{(x^3)} \oplus \frac{F[x]}{(x^2)}$.

(a) How many cyclic $F[x]$-module of order $p^3$ does $V$ have?
(b) How many cyclic $F[x]$-submodules of order $p^2$ does $V$ have?
(c) How many of the cyclic $F[x]$-submodules of $V$ of order $p^2$ are direct summands of $V$?

2.2 Let $V$ be an abelian group with generators $(v_1, v_2, v_3)$ that has the matrix $\begin{pmatrix} 2 & 0 & 6 \\ 6 & 12 & 0 \end{pmatrix}$ as a relation matrix. Express $V$ as a direct sum of cyclic groups.

2.3 Let $V$ be a 3-dimensional vector space over the field $\mathbb{Q}$ and let $T : V \to V$ be a linear operator having minimal polynomial $p(x) = (x - 2)^3$.

(a) How many one-dimensional $T$-invariant subspaces does $V$ have? Justify your answer.
(b) How many two-dimensional $T$-invariant subspaces does $V$ have? Justify your answer.

2.4 Let $A \in \mathbb{C}^{5 \times 5}$ be a diagonal matrix with exactly four distinct entries on its main diagonal.

(a) What is the dimension of the vector space over $\mathbb{C}$ of matrices that are polynomials in $A$.
(b) What is the dimension of the vector space over $\mathbb{C}$ of matrices $B \in \mathbb{C}^{5 \times 5}$ such that $AB = BA$?
(c) If $B \in \mathbb{C}^{5 \times 5}$ is a diagonal matrix with exactly four distinct entries on its main diagonal, is $B$ similar to a polynomial in $A$? Justify your answer.

2.5 Let $A$ and $B$ in $\mathbb{Q}^{n \times n}$ be $n \times n$ matrices and let $I \in \mathbb{Q}^{n \times n}$ denote the identity matrix. (a) State true or false and justify: If $A$ and $B$ are similar over an extension field $F$ of $\mathbb{Q}$, then $A$ and $B$ are similar over $\mathbb{Q}$.
(b) Let $M$ and $N$ be $n \times n$ matrices over the polynomial ring $\mathbb{Q}[x]$. Define “$M$ and $N$ are equivalent over $\mathbb{Q}[x]$”.
(c) State true or false and justify: If $\det(xI - A) = \det(xI - B)$, then $xI - A$ and $xI - B$ are equivalent.
(d) State true or false and justify: If $xI - A$ and $xI - B$ are equivalent over $\mathbb{Q}[x]$, then $A$ and $B$ are similar over $\mathbb{Q}$.

2.6 Classify up to similarity all matrices $A \in \mathbb{C}^{3 \times 3}$ such that $A^3 = I$, where $I$ is the identity matrix, i.e., write down all possibilities for the Jordan form of $A$.

2.7 Let $V$ be a finite-dimensional vector space over a field $F$, let $T : V \to V$ be a linear operator, and let $p(x) \in F[x]$ be the minimal polynomial of $T$. Assume that $p(x) = p_1^{r_1} \cdots p_k^{r_k}$, where $p_i \in F[x]$ are distinct monic irreducible polynomials, $i = 1, \ldots, k$, and $r_i$ are positive integers. Let $W_i = \{ v \in V | p_i(T)^r_i(v) = 0 \}$. Describe how to get linear operators $E_i : V \to V$, $i = 1, \ldots, k$, such that $E_i(V) = W_i$, $E_i^2 = E_i$ for each $i$, $E_iE_j = 0$ if $i \neq j$, and $E_1 + \ldots + E_k = I$ is the identity operator on $V$.

2.8 Let $V$ be a finite-dimensional vector space over an infinite field $F$ and let $T : V \to V$ be a linear operator. Give to $V$ the structure of a module over the polynomial ring $F[x]$ by defining $x\alpha = T(\alpha)$ for each $\alpha \in V$.

(a) Outline a proof that $V$ is a direct sum of cyclic $F[x]$-modules.
(b) In terms of the expression for $V$ as a direct sum of cyclic $F[x]$-modules, what are necessary and sufficient conditions in order that $V$ have only finitely many $T$-invariant $F$-subspaces? Explain.