f1) Give an example of an infinite dimensional vector space $V$ over $\mathbb{R}$, and linear operators $T$ and $S$ such that:
A) $S$ is onto, but not one-to-one.
B) $T$ is one-to-one, but not onto.

f2) State true or false and justify: if $V$ is a finite-dimensional vector space and $W_1$ and $W_2$ are subspaces of $V$ such that $V = W_1 \oplus W_2$, then for any subspace $W$ of $V$ we have $W = (W \cap W_1) \oplus (W \cap W_2)$.

False. Example: $<(1, 1)> \oplus <(0, 1)> = \mathbb{R}$, but for $W = (1, 0)$ we have $(<(1, 0) \cap <(1, 1)> ) \oplus (<(1, 0) \cap <(1, 1)>)$.

f3) Let $F$ be a field, take $m, n \in \mathbb{Z}^+$ and let $A \in F^{m \times n}$ be an $m \times n$ matrix.
A) Define “row space of $A$”
B) Define “col space of $A$”
C) Prove that the dimension of the row space of $A$ is equal to the dimension of the column space of $A$.

f4) Let $D$ be a principal ideal domain, let $n \in \mathbb{Z}^+$ and let $D^{(n)}$ denote a free $D$-module of rank $n$.
A) If $L$ is a submodule of $D^{(n)}$, prove that $L$ is a free $D$-module of rank $m \leq n$.
B) If $L$ is a proper submodule of $D^{(n)}$, prove or disprove that the rank of $L$ must be less than $n$.

f5) Let $D$ be a principal ideal domain and $V$ and $W$ denote free $D$-modules of rank 5 and 4 respectively. Assume that $\phi : V \rightarrow W$ is a $D$-module homomorphism, and that $B = \{v_1, \cdots, v_5\}$ is an ordered basis of $V$ and $B' = \{w_1, \cdots, w_4\}$ is an ordered basis of $W$.
A) Define what is meant by the coordinate vector of $v \in V$ with respect to the basis $B$.
B) Describe how to obtain a matrix $A \in D^{4 \times 5}$ so that left multiplication by $A$ on $D^5$ represents $\phi : V \rightarrow W$ with respect to $B$ and $B'$.
C) How does the matrix $A$ change if we change the basis $B$ by replacing $v_2$ by $v_2 + av_1$ for some $a \in D$?
D) How does the matrix $A$ change if we change the basis $B'$ by replacing $w_2$ by $w_2 + aw_1$ for some $a \in D$.

f6) Let $F$ be a subspace of the vector space $\mathbb{C}^{4 \times 4}$ such that for every $A, B \in F$ we have $AB = BA$.
A) Demonstrate with an example that it is possible for there to exist in $F$ five elements that are linearly independent over $\mathbb{C}$.
B) If there exists a matrix $A \in F$ having at least two distinct characteristic values, prove that $\text{dim}F \leq 4$.

f7) Let $V$ be a finite-dimensional vector space over an infinite field $F$ and let $T : V \rightarrow V$ be a linear operator. Give to $V$ the structure of a module over the polynomial ring $F[x]$ by defining $x \cdot \alpha = T(\alpha)$ for each $\alpha \in V$.
A) Outline a proof that $V$ is a direct sum of cyclic $F[x]$-modules.
B) In terms of the expression for $V$ as a direct sum of cyclic $F[x]$-modules, what are necessary and sufficient conditions in order that $V$ have only finitely many $T$-invariant subspaces? Justify your answer.

**f8)** Let $M$ be a module over the integral domain $D$. Recall that a submodule $N$ of $M$ is said to be pure iff whenever $y \in N$ and $a \in D$ are such that there exists $x \in M$ with $ax = y$, then there exists $z \in N$ with $az = y$.

A) If $N$ is a direct summand of $M$, prove that $N$ is pure in $M$.

B) For $x \in M$, let $\bar{x} = x + N$ denote the coset representing the image of $x$ in the quotient module $M/N$. If $N$ is a pure submodule of $M$, and $\text{ann} \bar{x} = \{a \in D : a\bar{x} = 0\}$ is the principal ideal $(d)$, prove that there exists $x' \in M$ such that $x + N = x' + N$ and $\text{ann} x' = \{a \in D : ax' = 0\}$ is the principal ideal $(d)$.

**f9)** Let $M$ be a finitely generated module over the polynomial ring $F[x]$, where $F$ is a field, and let $N$ be a pure submodule of $M$. Prove that there exists a submodule $L$ of $M$ such that $N + L = M$ and $N \cap L = 0$.

**f10)** Let $V$ be a finite-dimensional complex inner product space and let $T : V \to V$ be a linear operator. Prove that $T$ is self-adjoint iff $\langle T\alpha | \alpha \rangle$ is real for every $\alpha \in V$.

**f11)** Let $V$ be an abelian group generated by the elements $a, b, c$. Assume that $2a = 6b, 2b = 6c, 2c = 6a$, and that these three relations generate all the relations on $a, b, c$.

A) Write down a relation matrix for $V$.

B) Find generators $x, y, z$ for $V$ such that $V = \langle x \rangle \oplus \langle y \rangle \oplus \langle z \rangle$ is the direct sum of cyclic subgroups generated by $x, y, z$.

C) Express your generators $x, y, z$ in terms of $a, b, c$.

D) What is the order of $V$?

E) What is the order of the element $a$?

**f12)** Let $p$ be a prime integer and let $F = \mathbb{Z}/p\mathbb{Z}$ be the field with $p$ elements. Let $V$ be a vectorspace over $F$ and $T : V \to V$ a linear operator. Assume that $T$ has characteristic polynomial $x^3$ and minimal polynomial $x^2$.

A) Express $V$ as a direct sum of cyclic $F[x]$-modules.

B) How many 1-dimensional $T$-invariant subspaces does $V$ have?

C) How many of the 1-dimensional $T$-invariant subspaces of $V$ are direct sumands of $V$?

D) How many 2-dimensional $T$-invariant subspaces does $V$ have?

E) How many 2-dimensional $T$-invariant subspaces of $V$ are direct sumands of $V$?

**f13)** Let $V$ be a 5-dimensional vectorspace over the field $F$ and let $T : V \to V$ be a linear operator such that $\text{rank} \ T = 1$. List all polynomials $p(x) \in F[x]$ that are possibly the minimal polynomial of $T$. Explain.

**f14)** Let $F$ be a field.

A) What is the dimension of the vectorspace of all 3-linear functions $D : F^{3 \times 3} \to F$?

B) What is the dimension of the vectorspace of all 3-linear alternating functions $D : F^{3 \times 3} \to F$.
Let $V$ be a finite-dimensional vectorspace over a field $F$. Prove that a linear operator $T : V \to V$ has a cyclic vector iff every linear operator $S : V \to V$ that commutes with $T$ is a polynomial in $T$. 