Example 1. Let $a$ and $b$ be non-zero real numbers and compute

$$\int e^{ax} \cos bx \, dx.$$

**Solution:** Let $u = \cos bx$, $dv = e^{ax} \, dx$, and integrate by parts.

$$\int e^{ax} \cos bx \, dx = \frac{1}{a} e^{ax} \cos bx + \frac{b}{a} \int e^{ax} \sin bx \, dx. \quad (1)$$

To compute $\int e^{ax} \sin bx \, dx$, we again let $dv = e^{ax} \, dx$, but this time pick $u = \sin bx$. Then

$$\int e^{ax} \sin bx \, dx = \frac{1}{a} e^{ax} \sin bx - \frac{b}{a} \int e^{ax} \cos bx \, dx. \quad (2)$$

Plugging equation (2) into equation (1), we see that

$$\int e^{ax} \cos bx \, dx = \frac{1}{a} e^{ax} \cos bx + \frac{b}{a} \left( \frac{1}{a} e^{ax} \sin bx - \frac{b}{a} \int e^{ax} \cos bx \, dx \right)$$

$$\int e^{ax} \cos bx \, dx = \frac{1}{a} e^{ax} \cos bx + \frac{b}{a^2} e^{ax} \sin bx - \frac{b^2}{a^2} \int e^{ax} \cos bx \, dx$$

$$\left( 1 + \frac{b^2}{a^2} \right) \int e^{ax} \cos bx \, dx = \frac{1}{a} e^{ax} \cos bx + \frac{b}{a^2} e^{ax} \sin bx$$

$$\frac{a^2 + b^2}{a^2} \int e^{ax} \cos bx \, dx = \frac{1}{a^2} e^{ax} (a \cos bx + b \sin bx).$$

Finally, cancel the $a^2$ in the denominator of each side and divide by $a^2 + b^2$ (which is non-zero since $a$, $b \neq 0$) to arrive at

$$\int e^{ax} \cos bx \, dx = \frac{1}{a^2 + b^2} e^{ax} (a \cos bx + b \sin bx) + C$$

**Remark:** The most interesting case is the case where $a, b$ are both non-zero, as in the statement of the problem. If $a = 0$ but $b \neq 0$, then $e^{ax} = 1$, so the integral reduces to the much simpler $\int \cos bx \, dx = \frac{1}{b} \sin bx + C$. If $b = 0$ but $a \neq 0$, then $\cos bx = 1$, and the integral is $\int e^{ax} \, dx = \frac{1}{a} e^{ax} + C$. Finally, if $a = b = 0$, then the integral is $\int 1 \, dx = x + C$. 

Example 2. Let $a$ and $b$ be as in example 1. Compute

$$\int e^{ax} \sin bx \, dx.$$ 

Solution: Instead of plugging (2) into (1) as in example 1, plug (1) into (2) to get

$$\int e^{ax} \sin bx \, dx = \frac{1}{a} e^{ax} \sin bx - \frac{b}{a} \left( \frac{1}{a} e^{ax} \cos bx + \frac{b}{a} \int e^{ax} \sin bx \, dx \right)$$

$$\int e^{ax} \sin bx \, dx = \frac{1}{a} e^{ax} \sin bx - \frac{b}{a^2} e^{ax} \cos bx - \frac{b^2}{a^2} \int e^{ax} \sin bx \, dx.$$ 

After performing algebraic manipulations similar to example 1, we arrive at

$$\int e^{ax} \sin bx \, dx = \frac{1}{a^2 + b^2} e^{ax} (a \sin bx - b \cos bx) + C.$$ 

So what? Why suffer through the algebra above to arrive at such a messy answer? Well, now if you have to integrate something like $\int e^{-x} \cos 3x \, dx$ on an exam and you remember the formula, you can just plug in $a = -1$ and $b = 3$ into the answer from example 1 to get

$$\int e^{-x} \cos 3x \, dx = \frac{1}{10} e^{-x} (- \cos 3x + 3 \sin 3x) + C.$$ 

Remember: there’s no work required for a multiple choice exam, so you can skip right to the answer if you have a good enough memory!