Definition of Inner Product

Let $V$ be a real vector space. An **inner product** on $V$ is a function that assigns to each ordered pair of vectors $u$ and $v$ in $V$ a real number $(u, v)$ satisfying

(a) 
(b) 
(c) 
(d)

**Example 1.** In $\mathbb{R}^n$, the dot product of vectors

\[
\begin{bmatrix}
  u_1 \\
  u_2 \\
  \vdots \\
  u_n
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
  v_1 \\
  v_2 \\
  \vdots \\
  v_n
\end{bmatrix}
\]

is defined by

\[
(u, v) = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.
\]

Show that the dot product is an inner product.
Example 2. *Compute the (standard) inner product* \((u, v)\) *in* \(\mathbb{R}^4\).

(a) \(u = [1 \ 2 \ 3 \ 4], \ v = [0 \ 3 \ 2 \ 1]\)

(b) \(u = [1 \ 2 \ 3 \ 4], \ v = [-4 \ -3 \ 2 \ 1]\)

(c) \(u = [\frac{1}{2} \ \frac{1}{2} \ \frac{1}{2} \ \frac{1}{2}], \ v = [\frac{1}{2} \ \frac{1}{2} \ \frac{1}{2} \ \frac{1}{2}]\)

**Definition**

- A real vector space that has an inner product defined on it is called an

- In an inner product space, we define the length of a vector \(u\) by

In an inner product space, we have the following important inequalities:

- **Cauchy-Schwarz Inequality:**

- **Triangle Inequality:**

**Proof**
Example 3. Verify Cauchy-Schwarz Inequality and triangle inequality with

\[ u = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix} \]

Definition If \( V \) is an inner product space,
- we say the distance between two vectors \( u \) and \( v \) is
- we say two vectors \( u \) and \( v \) are orthogonal if

Definition Let \( V \) be an inner product space.
- A set \( S \) of vectors in \( V \) is called orthogonal if
- If, in addition,

Example 4. Are vectors \( u_1 = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \), \( u_2 = \begin{bmatrix} 0 & 1 \end{bmatrix} \), \( u_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \) orthogonal? Are they orthonormal?
Example 5. Are vectors $u_1 = \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$, $u_2 = \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$, $u_3 = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$, orthogonal?

Are they orthonormal?

Remark Let $S = \{u_1, u_2, \ldots, u_n\}$ be an orthogonal set of nonzero vectors in an inner product space $V$. Then $S$ is linearly _______