4.1 General Theory of $n\text{th}$ Order Linear Equations

An $n$th order linear differential equation is

$$P_n(t) \frac{d^n y}{dt^n} + P_{n-1}(t) \frac{d^{n-1} y}{dt^{n-1}} + \cdots + P_1(t) \frac{dy}{dt} + P_0(t) y = g(t)$$

where $P_0, P_1, \ldots, P_n, g$ are continuous functions on some interval $I$.

If $P_0$ is nowhere zero in the interval $I$, the equation can be written as

$$y^{(n)} + P_1(t) y^{(n-1)} + \cdots + P_{n-1}(t) y' + P_n(t) y = g(t)$$

The initial conditions are

$$y(t_0) = y_0, \quad y'(t_0) = y'_0, \quad \ldots, \quad y^{(n-1)}(t_0) = y^{(n-1)}_0$$

**Theorem (Existence and Uniqueness)** If $p_1, p_2, \ldots, p_n$ and $g$ are continuous on the open interval $I$, then there exists exactly one solution $y$ that satisfies $DE$ and $IC$.

The $n$-th order linear homogeneous equation is

$$y^{(n)} + P_1(t) y^{(n-1)} + \cdots + P_{n-1}(t) y' + P_n(t) y = 0$$

If $y_1, y_2, \ldots, y_n$ are solutions, then

$$y(t) = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

is also a solution.

The Wronskian of solutions $y_1, y_2, \ldots, y_n$ are

$$W = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y'_1 & y'_2 & \cdots & y'_n \\ \vdots & \vdots & \ddots & \vdots \\ y^{(n-1)}_1 & y^{(n-1)}_2 & \cdots & y^{(n-1)}_n \end{vmatrix}$$
Theorem  If the functions \( y_1, y_2, \cdots, y_n \) are solutions of the homogeneous equation and if
\[
\text{W}(y_1, \cdots, y_n)(t) \neq 0 \quad \text{for at least one point on } I,
\]
then every solution can be expressed as linear combination of \( y_1, \cdots, y_n \).

Remark A set of solutions \( y_1, y_2, \cdots, y_n \) whose Wronskian is nonzero are called a fundamental set of solutions.

Linear dependency of functions

The functions \( f_1, f_2, \cdots, f_n \) are said to be linearly dependent on an interval \( I \), if there exists a set of constants \( k_1, \cdots, k_n \) not all zero,
\[
k_1 f_1(t) + k_2 f_2(t) + \cdots + k_n f_n(t) = 0
\]
for all \( t \in I \). Otherwise, they are linearly independent.

Example 1. Determine whether the functions \( f_1(t) = 1 \), \( f_2(t) = t \), and \( f_3 = t^2 \) are linearly independent or dependent on the interval \( I : -\infty < t < \infty \).

\[
k_1 \cdot 1 + k_2 \cdot t + k_3 \cdot t^2 = 0
\]

At \( t = 0 \) \quad \quad \quad \quad \quad k_1 = 0

At \( t = 1 \) \quad \quad \quad \quad \quad k_1 + k_2 + k_3 = 0 \quad \Rightarrow \quad k_1 = 0

At \( t = -1 \) \quad \quad \quad \quad k_1 - k_2 + k_3 = 0 \quad \quad \quad \quad \quad k_3 = 0

linearly dependent.

Example 2. Determine whether the functions \( f_1(t) = 1 \), \( f_2(t) = 2 + t \), \( f_3 = 3 - t^2 \), and \( f_4 = 4t + t^2 \) are linearly independent or dependent on the interval \( I : -\infty < t < \infty \).

\[
k_1 + k_2 (2 + t) + k_3 (3 - t^2) + k_4 (4t + t^2) = 0
\]

\[
(1k_1 + 3k_2 + 3k_3 + k_4) t + (k_1 + 4k_2) t^2 = 0
\]

independently many

\[
k_4 = 1, \quad k_3 = 1, \quad k_2 = -1, \quad k_1 = 5
\]
Theorem

- If the functions $y_1, y_2, \ldots, y_n$ form a fundamental set of solutions
  
  then $y_1, \ldots, y_n$ are linearly independent.

- If $y_1, y_2, \ldots, y_n$ are linearly independent on some interval $I$

  then they form a fundamental set of solutions.

For the nonhomogeneous equation

$$
\frac{d^n y}{dt^n} + p_1(t) \frac{d^{n-1} y}{dt^{n-1}} + \cdots + p_{n-1}(t) \frac{dy}{dt} + p_n(t)y = g(t),
$$

the general solution is

$$
y(t) = c_1 y_1(t) + \cdots + c_n y_n(t) + \phi(t).
$$

4.2 Homogeneous Equations with Constant Coefficients

We consider the $n$th order linear homogeneous equation with constant coefficients:

$$
a_0 y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_{n-1} y'(t) + a_n y(t) = 0
$$

If $y = e^{rt}$ is a solution, then

$$
\begin{align*}
  &\left( a_0 r^n + a_1 r^{n-1} + \cdots + a_n \right) e^{rt} = 0 \\
\end{align*}
$$

Real and distinct roots

If the characteristic equation has $n$ real and distinct roots, then

$$
y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_n e^{r_n t}
$$

Example 3. Find the general solution of

$$
2y''' - 4y'' - 2y' + 4y = 0.
$$

$c_1$, $c_2$, $c_3$

$$
\begin{align*}
  &r^3 - 4r^2 - 2r + 4 = 0 \\
  &2r^3 (r - 2) - 2(r - 2) = 2(r^2 - 1)(r - 2) = 0 \\
\end{align*}
$$

$$
y = c_1 e^t + c_2 e^{-t} + c_3 e^{2t}
$$
Complex roots

If the characteristic equation has complex roots, then

\[ e^{rt} \cos(mt) \quad e^{rt} \sin(mt) \]

**Example 4.** Find the general solution of

\[ y^{(4)} - y = 0. \]

Repeated roots

If the characteristic equation has a repeated root \( r_1 \) and multiplicity is \( s \), then

\[ e^{r_1 t}, \quad t e^{r_1 t}, \quad \ldots \quad t^{s-1} e^{r_1 t} \]

If the characteristic equation has a pair of complex roots \( \lambda + i\mu \) repeated \( s \) times, then

\[ e^{t(\lambda + i\mu)} \quad t e^{t(\lambda + i\mu)} \quad t^{2} e^{t(\lambda + i\mu)} \]

**Example 5.** Find the general solution of

\[ y^{(4)} + 2y'' + y = 0. \]

\[ \sqrt{r^2 + 2r + 1} = 0 \]

\[ (r^2 + 1)^2 = 0 \]

\[ r = 1, \quad i, \quad -1, \quad -i \]

\[ y = C_1 \cos(t) + C_2 \sin(t) + C_3 t \cos(t) + C_4 t \sin(t) \]