1.1 Vector and integral identities

In this section we list some notation, vector and integral identities that are commonly used in the finite element formulation of the boundary-value problems in electromagnetics.

Let $u$ and $\mathbf{u}$ be a scalar and vector continuous differentiable functions defined in an open set $\Omega \subset \mathbb{R}^3$, respectively. Denote the gradient, divergence, and curl as

$$
\nabla u = \left( \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3} \right),
$$

$$
\nabla \cdot \mathbf{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \sum_{i=1}^3 \frac{\partial u_i}{\partial x_i},
$$

$$
\nabla \times \mathbf{u} = \left( \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right).
$$

Let $\Gamma \subset \mathbb{R}^3$ be a surface and $\mathbf{n}$ be the unit normal vector on $S$. Define the surface gradient and surface scalar curl on $\Gamma$ as

$$
\nabla_\Gamma u = \mathbf{n} \times (\nabla u \times \mathbf{n}), \quad \text{curl}_\Gamma \mathbf{u} = (\nabla \times \mathbf{u}) \cdot \mathbf{n}.
$$

We note that for a continuous differentiable function $u$ defined in a neighborhood of $\Gamma$ we have the relation

$$
\nabla u = \nabla_\Gamma u + \frac{\partial u}{\partial \mathbf{n}} \mathbf{n}
$$

between the spatial gradient and the surface gradient. Here $\frac{\partial u}{\partial \mathbf{n}}$ is the normal derivative on $\Gamma$. The surface divergence and surface vector curl are defined by duality:

$$
\int_{\Gamma} \text{div}_\Gamma \mathbf{u} \cdot v = -\int_{\Gamma} \mathbf{u} \cdot \nabla_\Gamma v \quad \text{for all } v \in C_0^\infty(\Gamma),
$$

$$
\int_{\Gamma} \text{curl}_\Gamma \mathbf{u} \cdot \mathbf{v} = \int_{\Gamma} \mathbf{u} \cdot \text{curl}_\Gamma \mathbf{v} \quad \text{for all } \mathbf{v} \in C_0^\infty(\Gamma).
$$

In the following, $f$ and $g$ denote scalars or scalar functions, and $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$
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denote vectors or vector functions.
\[
\begin{align*}
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) \\
\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &= (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} \\
\nabla(fg) &= f\nabla g + g\nabla f \\
\nabla \cdot (f\mathbf{u}) &= f\nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla f \\
\nabla \times (f\mathbf{u}) &= f\nabla \times \mathbf{u} - \mathbf{u} \times \nabla f \\
\nabla(\mathbf{u} \cdot \mathbf{v}) &= \mathbf{u} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{u}) + (\mathbf{u} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u} \\
\nabla \cdot (\mathbf{u} \times \mathbf{v}) &= \mathbf{v} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{v}) \\
\nabla \times (\mathbf{u} \times \mathbf{v}) &= \mathbf{u} \nabla \cdot \mathbf{v} - \mathbf{v} \nabla \cdot \mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u} \\
\n\nabla \cdot (\nabla f) &= \nabla^2 f = \Delta f \\
\n\nabla \times (\nabla f) &= 0 \\
\n\nabla \cdot (\nabla \times \mathbf{u}) &= 0
\end{align*}
\]

Gradient theorem
\[
\int \nabla u = \int_i u \mathbf{n}
\]

Divergence or Gauss theorem
\[
\int \nabla \cdot \mathbf{u} = \int_i \mathbf{u} \cdot \mathbf{n}
\]

Curl theorem
\[
\int \nabla \times \mathbf{u} = \int_i \mathbf{n} \times \mathbf{u}
\]

First scalar Green’s theorem
\[
\int \Omega \left[ u \nabla \cdot (a \nabla v) + a \nabla u \cdot \nabla v \right] = \int_i a \frac{\partial v}{\partial \mathbf{n}}
\]

Second scalar Green’s theorem
\[
\int \Omega \left[ u \nabla \cdot (a \nabla v) - v \nabla \cdot (a \nabla u) \right] = \int_i a \left( u \frac{\partial v}{\partial \mathbf{n}} - v \frac{\partial u}{\partial \mathbf{n}} \right)
\]

First vector Green’s theorem
\[
\int \Omega \left[ a(\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v}) - \mathbf{u} \cdot (\nabla \times a \nabla \times \mathbf{v}) \right] = \int_i a(\mathbf{u} \times \nabla \times \mathbf{v}) \cdot \mathbf{n}
\]

Second vector Green’s theorem
\[
\int \Omega \left[ \mathbf{v} \cdot (\nabla \times a \nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times a \nabla \times \mathbf{v}) \right] = \int_i a(\mathbf{u} \times \nabla \times \mathbf{v} - \mathbf{v} \times \nabla \times \mathbf{u}) \cdot \mathbf{n}
\]