Lectures on Grothendieck Duality
VII: The twisted inverse-image pseudofunctor.

Joseph Lipman
Purdue University
Department of Mathematics
lipman@math.purdue.edu

February 20, 2009
We have treated the right adjoint $\times$ of $\mathbf{R}(-)_*$ for quite general maps. But for non-proper maps this pseudofunctor may be of limited interest.

Grothendieck Duality is basically about a $\mathbf{D}^+_{\text{qc}}$-valued pseudofunctor $(-)!$ over the category of separated finite-type maps of noetherian schemes, agreeing with $\times$ on proper maps, but, unlike $\times$, agreeing with the usual inverse-image pseudofunctor $*$ on open immersions (more generally, on separated étale maps); and also compatible in a suitable sense with flat base change.

The existence and uniqueness (up to isomorphism) of this remarkable twisted inverse-image pseudofunctor is the first fundamental theorem to be discussed in this lecture; and its behavior vis-à-vis flat base change is the second.
1. Nagata’s compactification theorem.

2. Characterization of the twisted inverse image.

3. Base-change setups.

4. Flat base change for the twisted inverse image.

5. Enlargement of base-change setups.
Conventions

As before, we assume throughout that all schemes are noetherian,

For a scheme-map $h$, we write the abbreviation $h^*$ for $\mathbb{L}h^*$. 
A key point here is **Nagata’s compactification theorem**:

**Theorem**

Any finite-type separated map $f : X \to Y$ of noetherian schemes factors as $f = \overline{f}u$ with $\overline{f}$ proper and $u$ an open immersion.

**Remarks.**

1. Any such $\overline{f}$ is called a compactification of $f$.
2. For an easy example, think of the projective closure of an affine variety (or, more generally, an affine map).
   The problem is to paste local compactifications into a single global one.
3. For quasi-finite $f$, the theorem is essentially Zariski’s Main Theorem.
4. Nagata’s original paper appeared in 1962. The theorem is a hard one, and for a long time, his proof was not well-understood. But now there are several expositions, the most recent one, by Brian Conrad, having appeared in 2007 (J. Ramanujan Math. Soc., see references in reference notes.)
5. Suresh Nayak extended Nagata’s theorem, and hence the twisted inverse image, to essentially finite-type separated maps [arXiv:0809.1201].
Theorem

On the category $S_f$ of finite-type separated maps of noetherian schemes there is a $D^+_{qc}$-valued pseudofunctor $(-)^!$ that is uniquely determined, up to isomorphism, by the following three properties:

(i) The pseudofunctor $(-)^!$ restricts on the subcategory of proper maps to a right adjoint of the derived direct-image pseudofunctor.

(ii) The pseudofunctor $(-)^!$ restricts on the subcategory of étale maps to the usual inverse-image pseudofunctor $(-)^*$.

(iii) For any fiber square in $S_f$

$$
\begin{array}{ccc}
\bullet & \xrightarrow{v} & \bullet \\
g & \downarrow \sigma & \downarrow f \\
\bullet & \xrightarrow{u} & \bullet
\end{array}
$$

with $f$, $g$ proper, and $u$, $v$ étale, the base-change map $\beta_\sigma$ (previously defined) equals the natural composite isomorphism

$$
v^*f^! \sim (fv)^! \sim (ug)^! \sim g^!u^! = g^!u^*.
$$
In view of (i) and (ii), the obvious way to define $f^!$ is to compactify $f$, say $f = \overline{f} u$, and set

$$f^! := \overline{f}^! \circ u^*.$$  

The point is then to show, using flat base change for proper maps, that this definition is essentially independent of the chosen compactification, and that the result is pseudofunctorial and satisfies (iii).

The argument is based on a general method of Deligne for pasting together two pseudofunctors on subcategories of a given one.

To deal both with existence and (later) with base change for the twisted inverse image, a formalization of some basic features of base change will be useful.
A base-change setup $\mathcal{B}(S, P, F, !, *, (\beta_\sigma)_{\sigma \in \square})$ consists of the following data (a)–(d), subject to conditions (1)–(3):

(a) Subcategories $P$ and $F$ of a category $S$, each containing every object of $S$.

(b) Contravariant pseudofunctors $(-)^!$ on $P$ and $(-)^*$ on $F$, such that for all objects $X \in S$, the categories $X^!$ and $X^*$ coincide.

(c) A class $\square$ of commutative $S$-squares, called distinguished squares:

\[
\begin{array}{ccc}
\bullet & \xrightarrow{\nu} & \bullet \\
\downarrow{g} & \sigma & \downarrow{f} \\
\bullet & \xrightarrow{u} & \bullet \\
\end{array}
\]

\[ (f, g \in P; \; u, v \in F). \]

(d) For each distinguished $\sigma(u, g, f, v)$, an isomorphism of functors

\[ \beta_\sigma : \nu^*f! \sim \xrightarrow{\sim} g^!u*. \]
(1) If two commutative $S$-squares

$$
\begin{array}{ccc}
\bullet & \xrightarrow{v} & \bullet \\
g & \sigma & f \\
\bullet & \xrightarrow{u} & \bullet
\end{array}
$$

$$
\begin{array}{ccc}
\bullet & \xrightarrow{v_1} & \bullet \\
g_1 & \sigma_1 & f_1 \\
\bullet & \xrightarrow{u_1} & \bullet
\end{array}
$$

are isomorphic, then $\sigma$ is distinguished $\iff$ $\sigma_1$ is distinguished.

(2) For every $P$-map $f$ (resp. $F$-map $u$), the square

$$
\begin{array}{ccc}
\bullet & \xrightarrow{1} & \bullet \\
f & \sigma & f \\
\bullet & \xrightarrow{1} & \bullet
\end{array}
$$

resp. $\begin{array}{ccc}
\bullet & \xrightarrow{u} & \bullet \\
1 & \sigma & 1 \\
\bullet & \xrightarrow{u} & \bullet
\end{array}$

is distinguished.
(3) (Horizontal transitivity.) If the square $\sigma_0 = \sigma_2 \circ \sigma_1$ (with $g$ deleted)

```
  v1  \rightarrow
  \downarrow h
\sigma_1 \downarrow
  \rightarrow \sigma_2 \downarrow f
  u1  \rightarrow
```

as well as its constituents $\sigma_2$ and $\sigma_1$ are all distinguished, then
the corresponding natural diagram of functorial maps commutes:

```
(v_2v_1)^*f! \xrightarrow{\beta_{\sigma_0}} h^!(u_2u_1)^*
\downarrow \sim
v_1^*v_2^*f! \xrightarrow{v_1^* \beta_{\sigma_2}} v_1^*g^!u_2^* \xrightarrow{\beta_{\sigma_1}} h^!u_1^*u_2^*.
```

Similarly for transitivity vis-à-vis vertical juxtaposition of squares.
Examples

(A) (Pseudofunctors as base-change setups.)
Let $S$ be a category, take $P := F := S$, let $(-)^! = (-)^*$ be a contravariant pseudofunctor on $S$, let $\square := \{\text{all commutative squares in } S\}$, and for any such square $\sigma$, let

$$\beta_\sigma : v^*f^* \xrightarrow{\sim} (fv)^* = (ug)^* \xrightarrow{\sim} g^*u^*$$

be the isomorphism naturally associated with the pseudofunctor $(-)^*$. For this $(S, P, F, ^!, ^*, (\beta_\sigma)_{\sigma \in \square})$, conditions (1)–(3) are easily checked.

(B) $B(S_f, P, E, \times, ^*, (\beta_\sigma)_{\sigma \in \square})$, with:

- $S_f := \{\text{finite-type separated scheme-maps}\}$,
- $P \subset S_f := \{\text{proper maps}\}$, with $D^{+}_{qc}$-valued duality pseudofunctor $(-)^!$;
- $E \subset S_f := \{\text{étale maps}\}$, with $D^{+}_{qc}$-valued pseudofunctor $(-)^*$ where $u^* := Lu^*$ for any $E$-map $u$;
- $\square := \{\text{fiber squares } \sigma(u, g, f, v) \text{ with } f, g \in P, (u, v) \in E\}$;
- $\beta_\sigma : v^*f^\times \xrightarrow{\sim} g^\times u^* := \text{the corresponding base-change isomorphism.}$
To be able to apply Deligne’s pasting arguments to prove the theorem, we’ll need to enlarge this last $\mathcal{B}$ to a setup $\mathcal{B}(S_f, P, E, \times, *, (\beta'_{\sigma})_{\sigma \in \Box'})$ where $\Box'$ consists of all commutative $S_f$-squares $\sigma(u, g, f, v)$ with $f, g$ proper and $u, v$ étale.

That means we have to extend $\beta_{\sigma}$ to all $\sigma \in \Box'$, while maintaining transitivity.

In fact we will extend to an even larger class of squares:
In the category of schemes, an **admissible square** is a commutative square

\[
\begin{array}{ccc}
X' & \xrightarrow{v} & X \\
\downarrow^{g} & & \downarrow^{f} \\
Y' & \xrightarrow{u} & Y
\end{array}
\]

\((*)\)

such that in the associated diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{i} & X \times_YY' & \xrightarrow{q_1} & X \\
\downarrow^{q_2} & & \downarrow^{f} & & \\
Y' & \xrightarrow{u} & Y
\end{array}
\]

(where \(q_1, q_2\) are the projections, \(q_1i = v\) and \(q_2i = g\)) the map \(i\) is étale.

**Example:** Any commutative \((*)\) with \(u\) and \(v\) étale is admissible. That’s because a map which is étale remains so after any base change, and if \(qi\) and \(q\) are both étale then so is \(i\), see [EGA IV, §17].
4. Flat base change for the twisted inverse image

Theorem

- Let $S$ be the category of separated maps of noetherian schemes.
- Let $S_f \subset S$ be the subcategory of finite-type maps, and let $(-)^!$ be the $D^{+}_{qc}$-valued twisted inverse-image pseudofunctor on $S_f \subset S$.
- Let $F \subset S$ be the subcategory of flat maps, and let $(-)^*$ be the usual $D^{+}_{qc}$-valued inverse-image pseudofunctor on $F$.
- Let $\square$ be the class of admissible $S$-squares.

Then there is a unique base-change setup $B(S, S_f, F, ^!, ^*, (\beta_\sigma)_{\sigma \in \square})$ such that the following conditions hold for any $\sigma(u, g, f, v) \in \square$.

(i) If $\sigma$ is a fiber square with $f$ proper then $\beta_\sigma$ is the base-change isomorphism.

(ii) If $f$ —and hence $g$—is étale, so that $f^! = f^*$ and $g^! = g^*$, then $\beta_\sigma$ is the natural isomorphism $v^* f^* \sim g^* u^*$.

(iii) If $u$ —and hence $v$—is étale, so that $u^* = u^!$ and $v^* = v^!$, then $\beta_\sigma$ is the natural isomorphism $v^! f^! \sim g^! u^!$. 
In other words

The theorem says there is essentially one way to associate to each admissible square of noetherian schemes

```
• \xrightarrow{\nu} •
\downarrow \sigma \downarrow \quad \downarrow f
\bullet \quad \discretionary{}{}{}
\bullet
\downarrow g
\bullet \xrightarrow{u} •
```

(where \(f\) and \(g\) are finite-type separated maps, and \(u\) and \(\nu\) are flat), a functorial isomorphism \(\beta_\sigma : \nu^*f! \to g^!u^*\) that satisfies horizontal and vertical transitivity, and that for certain special admissible squares is the isomorphism specified by conditions (i)—(iii).
5. Enlargement of base-change setups

As has been indicated, the fundamental existence and base-change theorems for the twisted inverse image are proved via two kinds of abstract pasting theorems, one for pseudofunctors and one for base-change setups.

In general terms, pasting of data given on two subcategories of a category is done by making a fairly obvious construction which uses “compactifications” of maps—factorizations $f = \overline{f} u$ where $\overline{f}$ and $u$ are in the respective subcategories, and then checking via numerous commutative diagrams that the result is independent of the choice of compactification, and has all the desired properties.

For example, to construct $\beta_\sigma$ for an admissible square $\sigma(u, g, f, v)$, one compactifies $f$ to decompose $\sigma$ as follows:
where the two squares are fiber squares, \( \bar{f} \) and \( \bar{g} \) are proper, \( u_1 \) and \( u_2 \) are open immersions (hence étale), \( i \) is étale, and \( u, w, q \) are flat.

If the base-change theorem is to hold, then \( \beta_\sigma \) must be the natural composed isomorphism

\[
\begin{align*}
g ! u^* & \sim i ! u_2 ! \bar{g} ! u & \sim i ! u_2 ! w^* \bar{f} ! = \\
i^* u_2 ^* w^* \bar{f} ! & \sim i^* q^* u_1 ^* \bar{f} ! \sim v^* u_1 ^* \bar{f} ! \sim v^* f !.
\end{align*}
\]

Is this independent of the choice of compactification, and transitive?
The answer is, of course, yes. But for the most part, details of the proof are not very suitable for a lecture; they can be found in the reference notes.
Example of enlargement

Just to illustrate, in the rest of this lecture, we’ll sketch some techniques for enlarging base-change setups.

Recall the base-change setup $\mathcal{B} = \mathcal{B}(\mathcal{S}_f, \mathcal{P}, \mathcal{E}, \times, *, (\beta_\sigma)_{\sigma \in \square})$ with:

- $\mathcal{S}_f := \{\text{finite-type separated scheme-maps}\}$,
- $\mathcal{P} \subset \mathcal{S}_f := \{\text{proper maps}\}$, with $\mathcal{D}_{qc}^+$-valued duality pseudofunctor $!$;
- $\mathcal{E} \subset \mathcal{S}_f := \{\text{étale maps}\}$, with $\mathcal{D}_{qc}^+$-valued duality pseudofunctor $*$
  where $u^* := L u^*$ for any $\mathcal{E}$-map $u$;
- $\square := \{\text{fiber squares } \sigma(u, g, f, v) \text{ with } f, g \in \mathcal{P}, (u, v) \in \mathcal{E}\}$;
- $\beta_\sigma : v^* f^* \rightarrow g^* u^* := \text{the corresponding base-change isomorphism.}$

The problem is to extend $\beta_\sigma$ to a larger class of $\sigma$;
more precisely, to enlarge $\mathcal{B}$ to a setup $\mathcal{B}(\mathcal{S}_f, \mathcal{P}, \mathcal{E}, \times, *, (\beta'_\sigma)_{\sigma \in \square'})$ where $\square'$ consists of all commutative $\mathcal{S}_f$-squares $\sigma(u, g, f, v)$ with $f, g$ proper and $u, v$ étale.

We approach this problem axiomatically.
Special subcategories

For a setup $\mathcal{B}(\mathbf{S}, \mathbf{P}, \mathbf{E}, !, *, (\beta_\sigma)_{\sigma \in \Box})$, a subcategory $\mathbf{A} \subset \mathbf{S}$ is special if for any maps $i : X \to Y$ in $\mathbf{A}$, $g : X' \to X$ in $\mathbf{P}$, and $v : X' \to X$ in $\mathbf{E}$, $\Box$ contains the squares

\[
\begin{array}{ccc}
X' & \xrightarrow{1} & X' \\
g \downarrow & & ig \downarrow \\
X & \xrightarrow{i} & Y
\end{array} \quad \begin{array}{ccc}
X' & \xrightarrow{v} & X \\
1 \downarrow & & i \downarrow \\
X' & \xrightarrow{iv} & Y
\end{array}
\]

Example

For the preceding example, the category $\mathbf{A}$ whose maps are all the open-and-closed immersions is special.
Indeed, since $i$ is a monomorphism, the above squares are fiber squares.
A simple deduction

After fixing a special subcategory $\mathbf{A}$, we call its maps special. For any special map $i: X \to Y$, $\beta_i: i^! \sim i^*$ is defined to be the isomorphism $\beta_\tau$ associated to the distinguished square

$$
\begin{array}{ccc}
X & \xrightarrow{1} & X \\
\downarrow & & \downarrow \\
1 & \xrightarrow{\tau} & i \\
\downarrow & & \downarrow \\
X & \xrightarrow{i} & Y \\
\end{array}
$$

**Proposition**

Let $\mathbf{A}$ be a special subcategory of $\mathcal{B}(\mathbf{S}, \mathbf{P}, \mathbf{E}, 1, \ast, (\beta_\sigma)_{\sigma \in \square})$. Then the $\beta_i$ give a pseudofunctorial isomorphism of the restrictions of $1^!$ and $\ast$ to $\mathbf{A}$.

So, for instance, if $i$ is an open-and-closed immersion then $i^*$ is right-adjoint—pseudofunctorially—to $Ri_*$. (This fact can easily be shown directly; the foregoing relates it to base change.)
For pseudofunctoriality of the isomorphism $\beta_i$, apply (3) and (2) in the definition of setup to

\[
\begin{array}{ccc}
X & \xrightarrow{1} & X \\
\downarrow & & \downarrow \\
Y & \xrightarrow{1} & Y \\
\downarrow & & \downarrow \\
X & \xrightarrow{i} & Y & \xrightarrow{j} & Z
\end{array}
\]

to see that the left and right halves of the following diagram commute:

\[
\begin{array}{ccc}
(ji)^! & \xrightarrow{\cong} & (ji)^! \\
\downarrow & & \downarrow \\
(i^!j^!)(ji)^! & \xrightarrow{\cong} & (ji)^!
\end{array}
\]

\[
\begin{array}{ccc}
\cong & & \cong \\
\downarrow & & \downarrow \\
i^!j^! & \xrightarrow{\cong} & i^!j^*
\end{array}
\]

\[
\begin{array}{ccc}
\cong & & \cong \\
\downarrow & & \downarrow \\
(i^!j^!) & \xrightarrow{\cong} & i^*j^*
\end{array}
\]
Additional conditions

For the basic enlargement result, we impose additional mild conditions on $\mathcal{B}(S, P, E, !, *, (\beta_\sigma)_{\sigma \in \Box})$ and its special subcategory $A$. These conditions are easily verified for the (proper, ètale, fiber-square) setup we are most interested in at present.

(4) In the following $S$-diagrams, suppose that $u_1 \in E$ (resp. $f_1 \in P$).

\[
\begin{array}{c}
\bullet \xrightarrow{v_1} \bullet \xrightarrow{v_2} \bullet \\
\sigma_1 \downarrow g \downarrow \sigma_2 \downarrow f \\
\bullet \xrightarrow{u_1} \bullet \xrightarrow{u_2} \bullet
\end{array}
\]

In either diagram, if $\sigma_2$ is a fiber square and the composed square $\sigma_2 \sigma_1$ is in $\Box$, then $\sigma_1 \in \Box$. 
(5) If the $S$-square $\sigma(u, g, f, v)$

\[
\begin{array}{ccc}
X' & \xrightarrow{v} & X \\
\downarrow g & & \downarrow f \\
Y' & \xrightarrow{u} & Y
\end{array}
\]

is in $\Box$, and $u$ (resp. $f$) is special then so is $v$ (resp. $g$).

(6) If the above $\sigma(u, g, f, v)$ is in $\Box$ then so is any fiber square with the same $u$ and $f$,

\[
\begin{array}{ccc}
X'' & \longrightarrow & X \\
\downarrow & & \downarrow f \\
Y' & \xrightarrow{u} & Y
\end{array}
\]

and furthermore, the resulting map $X' \rightarrow X''$ is special.
Remark. Let $\mu : X' \rightarrow X''$ be an isomorphism and consider the following fiber squares, the first of which is, by (2), distinguished:

$$
\begin{array}{ccc}
X'' & \xrightarrow{1} & X'' \\
\downarrow & & \downarrow 1 \\
X'' & \xrightarrow{1} & X''
\end{array}
\quad
\begin{array}{ccc}
X' & \xrightarrow{\mu} & X'' \\
\downarrow & & \downarrow 1 \\
X'' & \xrightarrow{1} & X''
\end{array}
$$

From (6) it follows that $\mu$ is special. Thus every isomorphism is special.
Under the preceding assumptions on $\mathcal{B}(S, P, E, !, *, (\beta_\sigma)_{\sigma \in \Box})$ and $A$, there exists a unique base-change setup $\mathcal{B}' = \mathcal{B}'_A = \mathcal{B}(S, P, E, !, *, (\beta'_\sigma)_{\sigma \in \Box'})$ such that:

(i) A commutative square

$$
\begin{array}{ccc}
X' & \xrightarrow{v} & X \\
g \downarrow & & \downarrow f \\
Y' & \xrightarrow{u} & Y
\end{array}
$$

is in $\Box'$ if and only if there is a fiber square in $\Box$

$$
\begin{array}{ccc}
X'' & \rightarrow & X \\
\downarrow & & \downarrow f \\
Y' & \xrightarrow{u} & Y
\end{array}
$$

such that the resulting map $X' \rightarrow X''$ is special.

(Hence $\Box \subseteq \Box'$; and every fiber square in $\Box'$ is in $\Box$.)

(ii) For every $\sigma \in \Box \subseteq \Box'$ it holds that $\beta_\sigma = \beta'_\sigma$. 

Joseph Lipman (Purdue University) 
VII: Twisted inverse image. 
February 20, 2009 25 / 27
This Proposition does not yet suffice for our purposes, since for the proper, étale, fiber-square $B$, it only gives $\beta_{\sigma}$ for diagrams which decompose as

\[
\begin{array}{ccc}
\bullet & \xrightarrow{i} & \bullet \\
q_2 & \downarrow & \downarrow f \\
\bullet & \xrightarrow{u} & \bullet \\
\end{array}
\]

where the square is a fiber square and $i$ an open-and-closed immersion, whereas we need the result more generally for where $i$ is étale.

But the following Corollary provides a second enlargement to reach the desired situation:
In the Proposition, let $A'$ be a subcategory of $S$ such that for every map $i: X \rightarrow Y \in A'$ the diagonal map $\delta_i: X \rightarrow X \times_Y X$ is in $A$. Assume further that for any fiber square $\sigma_{v,f,g,u}$ in $S$, if $u$ (resp. $f$) is in $A'$ then so is $v$ (resp. $g$). Then :

(i) $A'$ is $B'$-special; and conditions (4)-(6) hold for $(B', A')$. Thus it is meaningful to set $B'' := (B')_{A'}$.

(ii) If a fiber square $\sigma(u,g,f,v)$ with $u \in A'$ is in $\Box$, then any commutative $\sigma_{v',f,g',u}$ with $v' \in A'$ and $g' \in P$ is $B''$-distinguished.

The proofs of the Proposition and its Corollary consist mainly of verifying formally the commutativity of a number of suitably chosen diagrams, some of them rather large.
Finally:

**Example**

The diagonal of a separated étale map is an open-and-closed immersion; and maps which are étale (resp. proper) remain so after arbitrary base change \([\text{EGA IV, §17}]\).

Therefore the category \(A'\) of proper étale maps satisfies the hypotheses of the Corollary with respect to the (proper, étale, fiber-square) setup \(B\) and its special subcategory \(A\) of open-and-closed maps.

The resulting setup \(B''\) is then the sought-after unique enlargement of \(B\) (i.e., the one where all commutative \(\sigma(u, g, f, v)\) with \(f, g\) proper and \(u, v\) étale are distinguished).

**To review:** Construct \(B'\), the enlargement of \(B\) via its special subcategory of open-and-closed immersions, then get \(B''\) as the enlargement of \(B'\) via its special subcategory of étale maps.