Section 1.1 (cont.): Difference Quotients

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A \textit{difference quotient} is an expression of the form

\[
\frac{f(x + h) - f(x)}{h}
\]

for a function \( f(x) \). As we will see shortly, difference quotients are essential to the definition of the derivative of a function. The derivative of a function will be the main topic for this class. In this lesson we will focus on using algebra to simplify difference quotients.
Consider the function \( f(x) = 3x + 5 \). Let’s compute the difference quotient of \( f \). Remember the difference quotient of \( f(x) \) is just a fraction whose numerator is \( f(x + h) - f(x) \) and whose denominator is \( h \). In this expression, \( h \) is a a separate variable, independent from \( x \). First, let’s compute the numerator.

\[
(x + h) - f(x) = [3(x + h) + 5] - (3x + 5)
= 3x + 3h + 5 - 3x - 5
= 3h
\]

Thus for the difference quotient, we have

\[
\frac{f(x + h) - f(x)}{h} = \frac{3h}{h} = 3
\]
As a general rule of thumb, you should always be able to cancel the $h$ in the denominator of the difference quotient. If you can’t cancel the $h$ in the denominator with an $h$ in the numerator, most likely there has been an error in computing $f(x + h) - f(x)$.

**Example**

Find the difference quotient of $f$.

- $f(x) = 1 - 7x$

\[
\frac{f(x + h) - f(x)}{h} = \frac{1 - 7(x + h) - (1 - 7x)}{h} = \frac{1 - 7x - 7h - 1 + 7x}{h} = \frac{-7h}{h} = -7
\]
\[ f(x) = 3x^2 + 6x - 1 \]

The idea here is the same as before, but the computation will be a little more complicated. First let’s compute \( f(x+h) \).

\[
\begin{align*}
 f(x + h) &= 3(x + h)^2 + 6(x + h) - 1 \\
 &= 3(x^2 + 2xh + h^2) + 6(x + h) - 1 \\
 &= 3x^2 + 6xh + 3h^2 + 6x + 6h - 1
\end{align*}
\]

Notice the \( 3x^2 \), the \( 6x \), and the \(-1\) will cancel with the terms from \( f(x) \). Therefore

\[
 f(x + h) - f(x) = 6xh + 3h^2 + 6h = h(6x + 3h + 6),
\]

and so

\[
\frac{f(x + h) - f(x)}{h} = \frac{h(6x + 3h + 6)}{h} = 6x + 3h + 6.
\]
Let’s finish with one more example. Again, the idea will be the same, but the algebra will be a little more involved.

\[ f(x) = \frac{1}{x-1} \]

\[
\frac{f(x + h) - f(x)}{h} = \frac{1}{x+h-1} - \frac{1}{x-1}
\]

To continue at this point, we must get a common denominator among the terms in the numerator. The easiest common denominator is just the product of the two denominators.

\[
\frac{\frac{1}{x+h-1} - \frac{1}{x-1}}{h} = \frac{\frac{x-1}{(x-1)(x+h-1)} - \frac{x+h-1}{(x-1)(x+h-1)}}{h}
\]

\[
= \frac{x-1-(x+h-1)}{(x-1)(x+h-1)}
\]

\[
= \frac{x-1-(x+h-1)}{h}
\]
\[
\frac{x-1-x-h+1}{(x-1)(x+h-1)} = \frac{-h}{(x-1)(x+h-1)} = \frac{h}{(x-1)(x+h-1)} \left(\frac{-1}{x-1}\right) = \frac{1}{(x-1)(x+h-1)}
\]