Lesson 20: Green’s Theorem

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In this lesson we examine a key theorem in vector calculus. It is in a sense the two dimensional analogue of the Fundamental Theorem of Line Integrals.

Consider a region $D$ enclosed by a simple closed curve $C$. We refer to the counterclockwise traversal of $C$ as the **positive orientation**, that is, if $C$ is given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$, then $D$ is always on the left as $\mathbf{r}(t)$ traverses $C$. 
Theorem (Green’s Theorem)

Let \( C \) be a positively oriented, piecewise-smooth, simple closed curve in the plane and let \( D \) be the region bounded by \( C \). If \( P \) and \( Q \) have continuous partial derivatives on an open region that contains \( D \), then

\[
\int_C P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA
\]

The theorem states that computing a certain line integral is the same as computing a certain double integral. In certain situations one or the other of these may be simpler, and so we can use this theorem to move between them.
We also use the notation $\oint_C P \, dx + Q \, dy$ for the line integral around a positively oriented closed curve $C$. In addition, the positively oriented boundary curve of a region $D$ is sometimes denoted $\partial D$. The statement of Green’s Theorem then becomes

$$\iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = \int_{\partial D} P \, dx + Q \, dy$$

Notice the similarities between this and the Fundamental Theorem of Calculus,

$$\int_a^b F'(x) \, dx = F(b) - F(a)$$

On the left side, we have the derivatives of our functions. On the right side, we look at values of the functions on the boundary of our region.
Example

Evaluate $\int_C x^4 \, dx + xy \, dy$, where $C$ is the triangular curve consisting of the line segments from $(0, 0)$ to $(1, 0)$, from $(1, 0)$ to $(0, 1)$, and from $(0, 1)$ to $(0, 0)$

(a) by directly computing the line integral,
(b) by applying Green’s Theorem.
Example

Use Green’s Theorem to evaluate $\oint_C x^2 y^2 \, dx + xy \, dy$, where $C$ consists of the arc of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ and the line segments from $(1,1)$ to $(0,1)$ and from $(0,1)$ to $(0,0)$. 
Example

Evaluate $\int_C (3y - e^{\sin x}) \, dx + (7x + \sqrt{y^4 + 1}) \, dy$, where $C$ is the circle $x^2 + y^2 = 9$. 
We can use Green’s Theorem to perform area computations. If \( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 \), then

\[
\iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = \iint_D 1 \, dA = A(D)
\]

The equation \( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 \) can result from several potential cases:

- \( P(x, y) = 0 \)
- \( P(x, y) = -y \)
- \( P(x, y) = -\frac{1}{2}y \)
- \( Q(x, y) = x \)
- \( Q(x, y) = 0 \)
- \( Q(x, y) = \frac{1}{2}x \)

Then by Green’s Theorem we have

\[
A = \oint_C x \, dy = -\oint_C y \, dx = \frac{1}{2} \oint_C x \, dy - y \, dx
\]
Example

Find the area enclosed by the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \).
The proof of Green’s Theorem given in the book applies only to \textit{simple} regions, that is, regions that are of type I and type II. Assuming only this, we can handle more general regions by breaking them up into simple regions.

\[
\iint_{D_1} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = \int_{C_1} P \, dx + Q \, dy + \int_{C_5} P \, dx + Q \, dy \\
+ \int_{C_2} P \, dx + Q \, dy + \int_{C_6} P \, dx + Q \, dy
\]
Example

Evaluate $\int_C y^2 \, dx + 3xy \, dy$, where $C$ is the boundary of the semiannular region $D$ in the upper half-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. 
Example

If $\mathbf{F}(x, y) = (-yi + xj)/(x^2 + y^2)$, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin.
Example

Use Green’s Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle \sqrt{x^2 + 1}, \tan^{-1} x \rangle$ and $C$ is the triangle from $(0, 0)$ to $(1, 1)$ to $(0, 1)$ to $(0, 0)$. 