Ch 1.2: Solutions of Some Differential Equations

• Recall the free fall and owl/mice differential equations:
  \[ v' = 9.8 - 0.2v, \quad p' = 0.5p - 450 \]

• These equations have the general form \( y' = ay - b \)
• We can use methods of calculus to solve differential equations of this form.

• Classify first order DE:

  • (1) Integrable equations
  • (2) Separable equations
  • (3) Linear equations
Integrable Equations

• (Examples)

(1) \[ y' = t^2 + 1 \]  
(2) \[ y' = \sin(t) \]  
(3) \[ e^{2t} y' = 5 \]  
(4) \[ y' = y - 2 \]

• Question: Can we use the same approach for the fourth equation?

• It is called a separable equation.
Separable Equations

• Question: Is there any way to transform the 4th equation into some equation close to integrable equation?

\[ y' = y - 2 \]

(a) an equilibrium solution: \( y(t) = 2 \)

(b) We assume that \( y \) is not an equilibrium solution.

• (Examples) Initial Value Problem (IVP)

\[ y' = 2y - 5, \quad y(0) = y_0 \]

(1) Find an equilibrium solution

(2) Find a general solution and the solution of the IVP.
Example 1: Mice and Owls  (1 of 3)

• To solve the differential equation

\[ \frac{dp}{dt} = 0.5p - 450 \]

we use methods of calculus, as follows.

\[ \frac{dp}{dt} = 0.5(p - 900) \quad \Rightarrow \quad \frac{dp}{p - 900} = 0.5 \quad \Rightarrow \quad \int \frac{dp}{p - 900} = \int 0.5 dt \]

\[ \Rightarrow \ln|p - 900| = 0.5t + C \quad \Rightarrow \quad |p - 900| = e^{0.5t+C} \]

\[ \Rightarrow \quad p - 900 = \pm e^{0.5t}e^C \quad \Rightarrow \quad p = 900 + ce^{0.5t}, \quad c = \pm e^C \]

• Thus the solution is

\[ p = 900 + ce^{0.5t} \]

where \( c \) is a constant.
Example 1: Integral Curves  (2 of 3)

• Thus we have infinitely many solutions to our equation,

\[ p' = 0.5p - 450 \implies p = 900 + ce^{0.5t}, \]

since \( c \) is an arbitrary constant.

• Graphs of solutions (integral curves) for several values of \( c \), and direction field for differential equation, are given below.

• Choosing \( c = 0 \), we obtain the equilibrium solution, while for \( c \neq 0 \), the solutions diverge from equilibrium solution.
Example 1: Initial Conditions  (3 of 3)

- A differential equation often has infinitely many solutions. If a point on the solution curve is known, such as an initial condition, then this determines a unique solution.

- In the mice/owl differential equation, suppose we know that the mice population starts out at 850. Then $p(0) = 850$, and

\[
p(t) = 900 + ce^{0.5t}
\]

\[
p(0) = 850 = 900 + ce^0
\]

\[
- 50 = c
\]

Solution:

\[
p(t) = 900 - 50e^{0.5t}
\]
Solution to General Equation

• To solve the general equation (: separable equation)
  \[ y' = ay - b \]
we use methods of calculus, as follows.

\[
\frac{dy}{dt} = a \left( y - \frac{b}{a} \right) \implies \frac{dy}{y - \frac{b}{a}} = a \implies \int \frac{dy}{y - \frac{b}{a}} = \int a \, dt
\]

\[ \implies \ln\left| y - \frac{b}{a} \right| = at + C \implies \left| y - \frac{b}{a} \right| = e^{at+C} \]

\[ \implies y - \frac{b}{a} = \pm e^{at}e^C \implies y = \frac{b}{a} + ce^{at}, \ c = \pm e^C \]

• Thus the general solution is
  \[ y = \frac{b}{a} + ce^{at}, \]
where \( c \) is a constant.
Next, we solve the initial value problem
\[ y' = ay - b, \quad y(0) = y_0 \]

From previous slide, the solution to differential equation is
\[ y = \frac{b}{a} + ce^{at} \]

Using the initial condition to solve for \( c \), we obtain
\[
 y(0) = y_0 = \frac{b}{a} + ce^0 \quad \Rightarrow \quad c = y_0 - \frac{b}{a}
\]

and hence the solution to the initial value problem is
\[
 y = \frac{b}{a} + \left[ y_0 - \frac{b}{a} \right] e^{at}
\]
Equilibrium Solution

• To find the equilibrium solution, set $y' = 0$ & solve for $y$:

$$y' = ay - b = 0 \implies y(t) = \frac{b}{a}$$

• From the previous slide, our solution to the initial value problem is:

$$y = \frac{b}{a} + \left[ y_0 - \frac{b}{a} \right] e^{at}$$

• Note the following solution behavior:
  - If $y_0 = b/a$, then $y$ is constant, with $y(t) = b/a$
  - If $y_0 > b/a$ and $a > 0$, then $y$ increases exponentially without bound
  - If $y_0 > b/a$ and $a < 0$, then $y$ decays exponentially to $b/a$
  - If $y_0 < b/a$ and $a > 0$, then $y$ decreases exponentially without bound
  - If $y_0 < b/a$ and $a < 0$, then $y$ increases asymptotically to $b/a$
Example 2: Free Fall Equation (1 of 3)

- Recall equation modeling free fall descent of 10 kg object, assuming an air resistance coefficient $\gamma = 2$ kg/sec:
  \[
  \frac{dv}{dt} = 9.8 - 0.2v
  \]

- Suppose object is dropped from 300 m. above ground.
  (a) Find velocity at any time $t$.
  (b) When does it hit ground and how fast will it be moving then?

- For part (a), we need to solve the initial value problem
  \[
  v' = 9.8 - 0.2v, \quad v(0) = 0
  \]

- Using result from previous slide, we have
  \[
  y = \frac{b}{a} + \left[ y_0 - \frac{b}{a} \right] e^{at} \Rightarrow v = \frac{9.8}{0.2} + \left[ 0 - \frac{9.8}{0.2} \right] e^{-2t} \Rightarrow v = 49(1 - e^{-2t})
  \]
Example 2: Graphs for Part (a)  

• The graph of the solution found in part (a), along with the direction field for the differential equation, is given below.

\[ v' = 9.8 - 0.2v, \quad v(0) = 0 \]

\[ v = 49 \left(1 - e^{-t/5}\right) \]
Example 2
Part (b): Time and Speed of Impact

• (Ex) Recall equation modeling free fall descent of 10 kg object, assuming an air resistance coefficient $\gamma = 2 \text{ kg/sec}$:
  Suppose object is dropped from 300 m. above ground.
  (a) Find velocity at any time $t$.
  (b) Next, given that the object is dropped from 300 m above ground, how long will it take to hit ground, and how fast will it be moving at impact?
Example 2
Part (b): Time and Speed of Impact (3 of 3)

(b) Next, given that the object is dropped from 300 m. above ground, how long will it take to hit ground, and how fast will it be moving at impact?

• Solution: Let $y(t) =$ distance object has fallen at time $t$. It follows from our solution $v(t)$ that

$$y'(t) = v(t) = 49 - 49e^{-t/5} \implies y(t) = 49t + 245e^{-t/5} + C$$

$$y(0) = 0 \implies C = -245 \implies y(t) = 49t + 245e^{-t/5} - 245$$

• Let $T$ be the time of impact. Then

$$s(T) = 49T + 245e^{-T/5} - 245 = 300$$

• Using a solver (root-finding s/w), $T \approx 10.51$ sec, hence

$$v(10.51) = 49\left(1 - e^{-0.2(10.51)}\right) \approx 43.01 \text{ m/sec}$$