Homework 3: Due Thursday, September 13

Reading: Chapters 3, 4

Problem 1: For each of the following operations on \( \mathbb{R} \), determine whether it is commutative, associative, has an identity, and inverses. Justify your answer (i.e. if it’s true, prove it; otherwise justify your answer.)

1. \( x \ast y = x + 2y - xy \)
2. \( x \ast y = |x - y| \)
3. \( x \ast y = \max\{x, y\} \) (i.e. the maximum of \( x \) and \( y \)).

Problem 2: Let \( G \) be a group, and \( g \in G \). Show that \((g^{-1})^{-1} = g\). If \( g, h \in G \), show that \((gh)^{-1} = h^{-1}g^{-1}\).

Problem 3: Let \( G \) be a group. Two elements \( g, h \in G \) are said to commute if \( gh = hg \). Show that if \( g \) and \( h \) commute, then so do their inverses.

Problem 4: If two elements \( a \) and \( b \) in a group \( G \) commute, show that \((ab)^n = a^nb^n\) for all \( n \in \mathbb{Z}\).

Problem 5: Let \( S \) be a set with an associative law of composition and with an identity element. Let \( G \) be the subset of \( S \) consisting of invertible elements (i.e. those \( s \in S \) for which there is an inverse under the given law of composition). Show that \( G \) is a group.

Problem 6: Write down the multiplication table for the group of symmetries of the square. Recall, as a set this is given by \( \{1, R, R^2, R^3, F, FR, FR^2, FR^3\} \). See problem set 2 for the notation used here.

Problem 7: Determine all integers \( n \) such that 2 has an inverse under multiplication modulo \( n \).

Problem 8: Show that the permutation group \( S_n \) is not abelian for \( n \geq 3 \) by explicitly exhibiting two elements in \( S_n \) which do not commute.