Homework 6: Due Thursday, October 11

**Reading:** Chapters 8,9,12,13.

**Problem 1** Express the following as a product of transpositions:

1. 
   
   (1) (416)(8235)

2. 
   
   (2) (123)(456)(1574)

**Problem 2** Let \( \alpha := (\alpha_1 \cdots \alpha_s) \in S_n \) be a cycle, and \( \pi \in S_n \). Show that \( \pi \alpha \pi^{-1} \) is the cycle \( (\pi(\alpha_1) \cdots \pi(\alpha_s)) \). Here we think of \( \pi \) as a permutation of \( \{1, \ldots, n\} \). Therefore, it makes sense to consider the elements \( \pi(\alpha_i) \in \{1, \ldots, n\} \).

**Problem 3** Given \( \alpha \) and \( \pi \) as in the previous exercise, the cycle \( \pi \alpha \pi^{-1} \) is called a conjugate of \( \alpha \). We say that a cycle \( \beta \) is a conjugate of \( \alpha \) if there is an element \( \pi \) such that \( \beta = \pi \alpha \pi^{-1} \). Note that this already implies that there exists a \( \pi' \) such that \( \alpha = \pi' \beta \pi'^{-1} \). Therefore, \( \alpha \) is a conjugate of \( \beta \) if and only if \( \beta \) is a conjugate of \( \alpha \). In this case, we simply say that \( \alpha \) and \( \beta \) are conjugates (or conjugate to one another). Use the last exercise to show that any two cycles of the same length are conjugates of each other.

**Problem 4:** Let \( \alpha, \beta \in S_n \). Show that \( \alpha \beta \) is even if an only if \( \alpha \) and \( \beta \) are both even or both odd.

**Problem 5:** Recall, given a group \( G \), its center \( Z(G) := \{ g \in G | gh = hg \forall h \in G \} \). You saw in Pbm. set 4, that this is a subgroup. Show that \( Z(S_n) = \{ e \} \) (i.e. it consists only of the identity permutation) for all \( n \geq 3 \).

**Problem 6:** Let \( G := \mathbb{R}_+ \) be the positive real numbers with group law given by usual multiplication. Is the function \( f(x) = x^2 \) from \( G \) to itself an isomorphism? If so, prove so. If not, demonstrate tha it’s not.

**Problem 7:** Let \( G \) be an abelian group, and \( A, B \subseteq G \) be two subgroups. Show that the set \( AB = \{ ab | a \in A, b \in B \} \subseteq G \) is also a subgroup.

**Problem 8:** Let \( G \) be a group such that \( g^2 = e \) for all \( g \in G \). Show that \( G \) is abelian.