Singular Limits in Polymer Stabilized Liquid Crystals

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Synopsis

We investigate equilibrium configurations for a polymer stabilized liquid crystal material subject to an applied magnetic field. The configurations are determined by energy minimization where the energies of the system include those of bulk, surface, and external field. The Euler-Lagrange equation is a nonlinear PDE with nonlinear boundary conditions defined on a perforated domain modeling the cross section of the liquid crystal-polymer fiber composite. We analyze the critical values for the external magnetic field representing Fredericks transitions and describe the equilibrium configurations under any magnitude of the external field. We also discuss the limit of the critical values and configurations as the number of polymer fibers approaches infinity. In the case where away from the boundary of the composite, the fibers are part of a periodic array, we prove that non-constant configurations develop order-one oscillations on the scale of the array’s period. Furthermore we determine the small-scale structure of the configurations as the period tends to zero.

1 Introduction

1.1 Polymer-liquid crystal composites

A nematic liquid crystal is a material that exists in an intermediate phase between liquid and solid. Its molecules are long thin rods that tend to align with one another.

*Partially supported by the National Science Foundation under Grant No.9971974
†Partially supported by the National Science Foundation under Grant No.9971713
The local average of the molecules’ principal axes near a point is identified with a unit vector $\mathbf{n}$ called the director. (See [5].) The resulting director field, $\mathbf{n}(\mathbf{x})$ for $\mathbf{x}$ ranging over the material body, defines the liquid crystal’s configuration. The configuration determines the optical characteristics of the liquid crystal and is very sensitive to an applied electric or magnetic field.

A Fredericks transition describes the onset of change, away from a uniform state, for a director configuration of a liquid crystal that is subjected to an applied field. The magnitude of the applied field for which a Fredericks transition occurs is called the critical value or threshold of the transition. Both the thresholds of the transitions and the configurations of the director fields are important for applications of liquid crystals in information display and processing.

New composite materials have been developed so as to stabilize particular configurations and to influence where critical values of applied fields occur. These are called polymer stabilized liquid crystals (PSLC). (See [3, 4, 6, 7, 8, 9, 13, 15] as well as references therein.) A composite is formed by embedding a polymer network in a liquid crystal matrix. In the absence of an applied field, the director field for a liquid crystal tends to align itself with the polymer network resulting in a preferred configuration. If an external field is applied to PSLC, it competes with the polymer network to determine the configuration in the liquid crystal. In this paper, assuming that a fixed polymer network has been assembled we examine a mathematical model obtaining qualitative descriptions for director configurations and their transitions under various magnitudes of the external magnetic field.

1.2 The model

We consider a polymer-liquid crystal composite in the shape of a long cylinder with its cross section in the $x_1x_2$ plane. The polymer network is modeled as a collection of fibers (rods) that extend through the cylinder in the $x_3$-direction. Here we ignore any cross-links in the network. Let $\mathcal{T} = \bigcup_{i=1}^{N} \mathcal{T}_i$ denote the cross section of the fibers, where we assume the $\{\mathcal{T}_i\}$ are mutually disjoint and that the $\mathcal{T}_i$ are simply connected open sets with $\partial \mathcal{T}_i$ of class $C^{2,\alpha}$ for some $0 < \alpha < 1$. Let $\Omega$ be a bounded domain in the $x_1x_2$ plane with a $C^{2,\alpha}$ boundary. Set $\mathcal{D} := \Omega \setminus \overline{\mathcal{T}}$. This set will denote the cross section of the liquid crystal matrix. Throughout the paper we assume that $\mathcal{D}$ is connected and that $\Omega \cap \mathcal{T}_i \neq \emptyset$ for each $i$. See Figure 1.
Consider a cylinder whose height, $L$, is large relative to the diameter of its cross section. Set $\text{diam}(\Omega) = W$. We assume that the cylinder occupies $(\Omega \cup \mathcal{T}) \times (0, L)$ with $L >> W$. Set $D = D \times (0, L)$. This body denotes the perforated cylindrical region occupied by the liquid crystal. Furthermore let $S = \partial D \setminus \partial \Omega$, $S_e = \partial D \setminus S$, $S = S \times (0, L)$ and $S_e = S_e \times (0, L)$. Then $S$ denotes the interface between the liquid crystal and the polymer fibers and $S_e$ represents the lateral surface between the liquid crystal and the exterior of the composite. See Figure 2.

We use energy minimization to analyze configurations for the liquid crystal and its Fredericks transitions under an external magnetic field. There are three types of relevant energies to take into account. These are the bulk energy of the liquid crystal, the energy due to the application of the external magnetic field and the surface energies on the interface between liquid crystal and polymer $S$, and on the outer surface $S_e$.

Let $n = (n_1, n_2, n_3)$ be the unit vector field denoting the director of the liquid crystal. The Frank-Oseen theory (see [5]) defines the elastic bulk energy density of a nematic liquid crystal as

$$K_1(\text{div } n)^2 + K_2(n \cdot \text{curl } n)^2 + K_3|n \times \text{curl } n|^2 + (K_2 + K_4)(\text{tr}(\nabla n)^2 - (\text{div } n)^2)$$
where $K_1$, $K_2$, $K_3$, and $K_4$ are material constants for the liquid crystal depending on the temperature. If $K_1 = K_2 = K_3 = K > 0$ and $K_4 = 0$, this gives the one constant approximation case where the energy density becomes

$$K|\nabla n|^2.$$

An external magnetic field can change the configuration of the liquid crystal. Under a uniform magnetic field the energy density is

$$m(n) = -\chi H^2(h \cdot n)^2$$

where $\chi > 0$ is a material constant, $H$ is the magnitude of the field and $h$ is a unit vector representing the direction of the field. In this paper we take $h$ perpendicular to the fibers. We shall assume that

$$h = e_2$$

so that the field tends to make the directors turn in the $x_2$-direction.

The interaction between the nematic liquid crystal and the polymer network is described by a weak anchoring condition on the interface expressed as a surface energy. We are interested in the effect of having a large contact area between the liquid crystal and the polymer on the qualitative features of the Fredericks transition.

On the interface between the liquid crystal and the polymer, $S$, we consider a weak anchoring condition with the Rapini-Papoular surface energy density

$$\epsilon w(1 - (n \cdot q)^2)$$

where $w > 0$ is a material constant representing the strength of the surface energy and $\epsilon$ is a dimensionless scaling factor for which $\epsilon^{-1}$ is proportional to the normalized inter-facial surface area $\mathcal{H}^2(S)/WL = \mathcal{H}^1(S)/W$. (Here $\mathcal{H}^n(S)$ is the $n$-dimensional Hausdorff measure of $S$.) The unit vector $q$ is parallel to the easy axis of the liquid crystal defined on the surface of the polymer. In this paper we shall assume $q = e_3$, i.e., in the direction of the polymer fibers.

**Remark.** The scaling factor $\epsilon$ is distinguished so that the surface energy remains bounded when considering a family of networks having larger and larger surface area, $\epsilon^{-1} \to \infty$. Without this factor the weak anchoring condition, (1.1), becomes a strong anchoring condition ($n = \pm q$ on $S$) as $\epsilon \to 0$.

On the outside boundary, $S_e$, we impose another weak anchoring condition with surface energy density

$$\beta w(1 - (n \cdot q)^2),$$
where $\beta > 0$ is another dimensionless scaling factor.

Since $L \gg W$ the surface areas of the top and bottom of the cylinder segment are small relative to that of $S_e$. As a result, we ignore contributions to the surface energy due to them.

For the one-constant approximation model then, we consider configurations that minimize the following energy

$$E(\mathbf{n}) = \int_D K|\nabla \mathbf{n}|^2 dx_1 dx_2 dx_3 + \int_{S_e} \epsilon w(1 - (\mathbf{n} \cdot \mathbf{e}_3)^2) d\mathcal{H}^1 dx_3$$

$$+ \int_{S_e} \beta w(1 - (\mathbf{n} \cdot \mathbf{e}_3)^2) d\mathcal{H}^1 dx_3 - \int_D \chi H^2(\mathbf{e}_2 \cdot \mathbf{n})^2 dx_1 dx_2 dx_3$$

where we seek solutions in $H^1(D; \mathbb{S}^2)$. Our energy is based on a fibril model given in [19]. The specific structure we consider is an idealization of that observed in [10].

1.3 Statement of main results

In Section 2 the problem is made dimensionless by introducing the transformations

$$(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \left(\frac{x_1}{W}, \frac{x_2}{W}, \frac{x_3}{W}\right)$$

and $\bar{\mathbf{n}}(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \mathbf{n}(x_1, x_2, x_3)$. Using these (1.2) becomes

$$W \int_D K|\nabla \bar{\mathbf{n}}|^2 d\bar{x}_1 d\bar{x}_2 d\bar{x}_3 + W^2 \int_{S_e} \epsilon w(1 - (\bar{\mathbf{n}} \cdot \mathbf{e}_3)^2) d\mathcal{H}^1 d\bar{x}_3$$

$$+ W^2 \int_{S_e} \beta w(1 - (\bar{\mathbf{n}} \cdot \mathbf{e}_3)^2) d\mathcal{H}^1 d\bar{x}_3 - W^3 \int_D \chi H^2(\mathbf{e}_2 \cdot \bar{\mathbf{n}})^2 d\bar{x}_1 d\bar{x}_2 d\bar{x}_3$$

where $\text{diam}(\bar{\Omega}) = 1$. Suppressing the tilde, we show that minimizers take the form

$$\mathbf{n} = \mathbf{n}(x_1, x_2) = (0, \sin \theta, \cos \theta)$$

where $\theta(x_1, x_2) \in H^1(D)$ is the angle between $\mathbf{e}_3$ and $\mathbf{n}(x_1, x_2)$.

We will take $K/w = \epsilon^2 W$ and focus on the case of $\epsilon$ small, physically indicating a stronger surface energy relative to the bulk liquid crystal energy. Setting $M = W\chi H^2/w$ we arrive at an unconstrained, scalar minimum problem on the cross section. Minimize

$$\mathcal{E}(\theta) = \epsilon^2 \int_D |\nabla \theta|^2 dx_1 dx_2 + \epsilon \int_{S_e} \sin^2 \theta d\mathcal{H}^1$$

$$+ \beta \int_{S_e} \sin^2 \theta d\mathcal{H}^1 - M \int_D \sin^2 \theta dx_1 dx_2.$$
for $\theta$ in $H^1(\mathcal{D})$. The associated equilibrium problem is

$$
\begin{align*}
\epsilon^2 \Delta \theta + M \sin \theta \cos \theta &= 0 \quad \text{in } \mathcal{D}, \\
\epsilon \frac{\partial \theta}{\partial \nu} + \sin \theta \cos \theta &= 0 \quad \text{on } \mathcal{S}, \\
\epsilon^2 \frac{\partial \theta}{\partial \nu} + \beta \sin \theta \cos \theta &= 0 \quad \text{on } \mathcal{S}_e.
\end{align*}
$$

(1.4)

We show that it suffices to consider solutions valued in $[0, \pi/2]$, and that these weak solutions are classical away from $\mathcal{S} \cap \mathcal{S}_e$. Note that $\theta \equiv 0$ and $\theta \equiv \pi/2$ are solutions to (1.4). In Section 3, for the case where $\mathcal{S} \cap \mathcal{S}_e = \emptyset$, we establish a comparison principle for solutions to (1.4). Using this we prove that $\theta \equiv 0$ is the unique minimizer if the strength of the magnetic field, $M$, is small and $\theta \equiv \pi/2$ is the unique minimizer if $M$ is large. In particular, we prove that there exist two values for $M$, $\mu < \overline{\mu}$, at which the second variation of $\mathcal{E}$ becomes degenerately stable for $\theta \equiv 0$ (with $M = \mu$) and $\theta \equiv \pi/2$ (with $M = \overline{\mu}$). These are the critical values of the two Fredericks transitions for this problem. We show that $\theta \equiv 0$ is the unique minimizer for (1.3) if $M \leq \mu$ and that $\theta \equiv \pi/2$ is the unique minimizer for (1.3) if $\overline{\mu} \leq M$. If $\mu < M < \overline{\mu}$ we prove that there exists a unique minimizer and that it is increasing with $M$. These solutions then describe the stable, quasi-static transition from the states $\theta \equiv 0$ to $\theta \equiv \pi/2$ as $M$ increases from $\mu$ to $\overline{\mu}$.

A principal objective in this work is to analyze the limit of critical values and configurations of minimizers as the number of polymer fibers goes to infinity, which corresponds to finer and finer distributions in the polymer networks. If the fibers are distributed uniformly, as part of a periodic array, we compare minimizers for (1.3) to those for a corresponding cell problem. To describe this we introduce a fundamental unit cell

$$Y := \{(x_1, x_2) \mid -\frac{1}{2} < x_1, x_2 < \frac{1}{2}\}.$$  

We let

$$T \subset \subset Y$$

represent the cross section of a fundamental polymer rod where $T$ is a simply connected open set with a $C^{2,\alpha}$ boundary. Define $Y^* := Y \setminus \overline{T}$ as the region in $Y$ occupied by the liquid crystal. For each $\epsilon > 0$, we distinguish a cross section homothetic to $T$,

$$T(\epsilon) := \{\epsilon x \mid x \in T\}.$$  

Then

$$\{(\overline{T}(\epsilon) + \epsilon z \mid z \in Z \times Z\}$$

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where \( Z = \{ \ldots, -1, 0, 1, \ldots \} \) is an \( \epsilon \)-periodic array of cross sections, and

\[
\mathcal{Y}(\epsilon) := \mathbb{R}^2 \setminus \bigcup_{z \in Z \times Z} (T(\epsilon) + \epsilon z)
\]

is the cross section of the \( \epsilon \)-periodic liquid crystal matrix in \( \mathbb{R}^2 \).

Let

\[
\mathcal{P}(Y^*) = \{ w \in H^1(Y^*) : w(-\frac{1}{2}, x_2) = w(\frac{1}{2}, x_2) \text{ for } -\frac{1}{2} \leq x_2 \leq \frac{1}{2} \text{ and } w(x_1, -\frac{1}{2}) = w(x_1, \frac{1}{2}) \text{ for } -\frac{1}{2} \leq x_1 \leq \frac{1}{2} \}
\]

and define

\[
\mathcal{E}_p(w) = \int_{Y^*} (|\nabla w|^2 - M \sin^2(w))dx_1dx_2 + \int_{\partial T} \sin^2(w)d\mathcal{H}^1.
\]

In Section 4.2 we prove that \( \mathcal{E}_p \) has a unique minimizer, valued in \([0, \pi/2]\), in the class of periodic functions \( \mathcal{P}(Y^*) \), which we denote as \( \varphi(x, M) \). Furthermore we show that there exist two critical values, \( \underline{\lambda} \) and \( \bar{\lambda} \), so that if \( M \leq \underline{\lambda} \) then \( \varphi(x, M) \equiv 0 \), if \( \underline{\lambda} \leq M \) then \( \varphi(x, M) \equiv \pi/2 \), and such that \( \varphi(x, M) \) is increasing in \( M \) if \( \underline{\lambda} < M < \bar{\lambda} \). Let \( \tilde{\varphi}(x, M) \) be the 1-periodic extension of \( \varphi(x, M) \) to \( \mathcal{Y}(1) \) and define

\[
\tilde{\varphi}_\epsilon(x, M) = \tilde{\varphi}(x/\epsilon, M)
\]

for \( x \) in the cross section of the \( \epsilon \)-periodic matrix \( \mathcal{Y}(\epsilon) \).

Set \( D = D(\epsilon) \) and consider the case where \( D(\epsilon) = \Omega \cap \mathcal{Y}(\epsilon) \). We prove that \( \lim_{\epsilon \to 0} \mu(\epsilon) = \underline{\lambda} \). (See Theorem 4.6.) Thus the lower critical value for the cell problem is a good approximation to \( \mu(\epsilon) \) if fine periodic networks are considered. However though, the upper threshold is influenced by a boundary layer near the exterior boundary \( S_\epsilon \) and tends to infinity, \( \lim_{\epsilon \to 0} \overline{\mu}(\epsilon) = \infty \). (See Theorem 4.7.) Thus the cell problem does not give an accurate estimate for the transition’s critical value at \( \theta \equiv \pi/2 \).

In Section 4.4 we prove our main result for minimizers, \( \theta_\epsilon(x, M) \), to (1.3). Assume for each compact set \( K \subset \Omega \) there is an \( \epsilon_0(K) \) so that \( D(\epsilon) \cap K = \mathcal{Y}(\epsilon) \cap K \) for all \( \epsilon < \epsilon_0 \). We show that

\[
(1.5) \quad \theta_\epsilon(x, M) - \tilde{\varphi}_\epsilon(x, M) \to 0 \text{ as } \epsilon \to 0,
\]

uniformly for \( x \in K \cap \overline{D(\epsilon)} \). Thus the minimizer of the corresponding cell problem provides a uniform approximation to minimizers for the composite system, away from
the composite’s exterior. (See Theorem 4.10.) Notice that the minimizers \( \theta_\epsilon(x, M) \) do not converge pointwise but develop oscillations as \( \epsilon \to 0 \). Also notice that even though a complete transition to \( \theta \equiv \pi/2 \) can not occur until \( M \geq \overline{\mu}(\epsilon) > \overline{\lambda} \), in fact \( \theta_\epsilon(x, M) \) is uniformly close to \( \pi/2 \) away from \( S_\epsilon \) for all \( M \geq \overline{\lambda} \) and \( \epsilon \) sufficiently small. This is due to \( \tilde{\varphi}_\epsilon(x, M) \equiv \pi/2 \) if \( M \geq \overline{\lambda} \) and (1.5).

Lastly, we comment on previous and related work. Prior studies for the analysis of equilibrium configurations focused on pure liquid crystal in special settings. These led to one-dimensional mathematical problems (e.g., nonlinear ODE with various boundary conditions). See Virga [17] for discussion in one-dimensional settings. However though, for composite materials, a one-dimensional model is not suitable. In the area of two-dimensional settings, Wang [18] analyzed the existence of Fredericks transitions for a “light-nematic” system in a rectangle domain without any holes. The critical value problems in composite systems that we are considering also have connections with eigenvalue problems in perforated domains (See Vanninathan [16] and Kaizu [12]).

2 Existence results

Consider the minimum problem for the one-constant approximation model given in (1.2). The problem is made dimensionless by writing

\[
\tilde{E}^{(3)}(\mathbf{n}) = \frac{1}{W} \int_D |\nabla \mathbf{n}|^2 \, dx_1dx_2dx_3 + \frac{\epsilon}{Wb} \int_S (1 - (\mathbf{n} \cdot \mathbf{e}_3)^2) \, dH^1 \, dx_3 \\
+ \frac{\beta}{Wb} \int_{S_c} (1 - (\mathbf{n} \cdot \mathbf{e}_3)^2) \, dH^1 \, dx_3 - \frac{1}{W\xi^2} \int_D (\mathbf{e}_2 \cdot \mathbf{n})^2 \, dx_1dx_2dx_3
\]

where \( b = K/w \) is the extrapolation length and \( \xi = \sqrt{\frac{K}{\chi H^2}} \) is the magnetic coherence length (see [5]). We let \( \tilde{E}^{(3)} = E^{(3)}/KW \), \( (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = \left( \frac{x_1}{W}, \frac{x_2}{W}, \frac{x_3}{W} \right) \), \( \tilde{\mathbf{n}}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = \mathbf{n}(x_1, x_2, x_3), b' = b/W, \xi' = \xi/W \) and suppress the tilde. The non dimensional version of the total energy becomes

\[
E^{(3)}(\mathbf{n}) = \int_D |\nabla \mathbf{n}|^2 \, dx_1dx_2dx_3 + \frac{\epsilon}{b'} \int_S (1 - (\mathbf{n} \cdot \mathbf{e}_3)^2) \, dH^1 \, dx_3 \\
+ \frac{\beta}{b'} \int_{S_c} (1 - (\mathbf{n} \cdot \mathbf{e}_3)^2) \, dH^1 \, dx_3 - \frac{1}{\xi'^2} \int_D (\mathbf{e}_2 \cdot \mathbf{n})^2 \, dx_1dx_2dx_3
\]

where the regions and boundaries have been scaled accordingly. In particular, now \( D = \Omega \setminus \mathcal{T} \times (0, \frac{L}{W}), \text{diam} (\Omega) = 1 \) and \( \frac{L}{W} >> 1 \).
We seek minimizers of \( E^{(3)} \) in \( \mathcal{F}^{(3)} = \{ \mathbf{n}(x_1, x_2, x_3) \in H^1(D; S^2) \} \). In the following theorem we use the same ideas as in [2] to show that a minimizer does not depend on \( x_3 \).

Let
\[
\mathcal{F} = \{ \mathbf{n}(x_1, x_2) \in H^1(D; S^2) \},
\]
\[
E(n) = \int_D |\nabla_{x_1 x_2} \mathbf{n}|^2 dx_1 dx_2 + \frac{\epsilon}{b'} \int_S (1 - n_3^2) dH^1
\]
\[
+ \frac{\beta}{b'} \int_{S_e} (1 - n_3^2) dH^1 - \frac{1}{\xi^2} \int_D n_3^2 dx_1 dx_2.
\]

**Theorem 2.1** A function \( n \) minimizes \( E^{(3)} \) in \( \mathcal{F}^{(3)} \) if and only if \( n = n(x_1, x_2) \in \mathcal{F} \) and minimizes \( E \) in \( \mathcal{F} \). Moreover minimizers for \( E \) in \( \mathcal{F} \) exist and are analytic in \( D \).

**Proof:** Minimizers for \( E \) in \( \mathcal{F} \) exist and are classical (analytic) in \( D \) (See [11]). Let \( n_0 \in \mathcal{F} \) such that \( E(n_0) = \min_{n \in \mathcal{F}} E(n) \) and \( \tilde{n} \) any element in \( \mathcal{F}^{(3)} \). Then
\[
E^{(3)}(\tilde{n}) = \int_0^{b'} \mathbb{W}(\tilde{n})(x_3) dx_3 + \int_0^{b'} \int_D \left| \frac{\partial \tilde{n}}{\partial x_3} \right|^2 dx_1 dx_2 dx_3
\]
\[
\geq E(n_0) \frac{L}{\mathbb{W}}.
\]
Thus \( n_0 \) minimizes \( E^{(3)} \) in \( \mathcal{F}^{(3)} \) and any minimizer of \( E^{(3)} \) in \( \mathcal{F}^{(3)} \) is independent of \( x_3 \).

Theorem 2.1 asserts that we only need to discuss the two dimensional problem of finding minimizers for \( E \) in \( \mathcal{F} \).

If there is no external magnetic field, i.e., \( \xi = \infty \), then it is obvious that \( E(n) \geq 0 \) and \( E(n) = 0 \) if and only if \( n = \pm e_3 \). These are trivial minimizers. Our next lemma shows that the components of a minimizer are either identically zero or strictly of one sign.

**Lemma 2.2** If \( n \) is a minimizer for \( E \) in \( \mathcal{F} \) and \( n_i \) is a component of \( n \) then either \( n_i \equiv 0 \) in \( D \) or \( n_i(x) \neq 0 \) for all \( x \in D \).

**Proof:** Consider \( \tilde{n} \in \mathcal{F} \) where \( \tilde{n} \) is obtained from \( n \) by replacing \( n_i \) with \( |n_i| \). Then we see \( \tilde{n} \) is also a minimizer. Thus it is without loss of generality to assume \( n_i \geq 0 \). Since \( n \) is regular it satisfies an equilibrium equation in \( D \) of the form \( \Delta n_j = \lambda_j n_j \) in \( D \) for \( 1 \leq j \leq 3 \) where \( \lambda_j = \lambda_j(x) \) is a smooth function (see [17]). From the maximum principle however nonnegative solutions are either identically zero or never vanish. ■
We see that it is without loss of generality to only discuss minimizers with non-negative components. The next theorem however shows that the $n_1$ component of a minimizer is always zero. Recall $e_1$ is perpendicular to both the easy axis of the weak anchoring condition, $e_3$, and the direction of the applied field, $e_2$.

**Theorem 2.3** If $\mathbf{n} = (n_1, n_2, n_3)$ is a minimizer of $\mathbb{E}$ in $\mathcal{F}$ then $n_1 \equiv 0$ in $\mathcal{D}$.

**Proof:** We argue by contradiction. In light of the previous lemma then we can assume that $n_1 > 0$ in $\mathcal{D}$. Consider the unit vector field

$$\mathbf{n} = (0, \mathbf{n}_2, n_3)$$

where $\mathbf{n}_2 = (n_1^2 + n_2^2)^{1/2} > 0$. Then $\mathbf{n} \in \mathcal{F}$ and

$$\mathbb{E}(\mathbf{n}) - \mathbb{E}(\mathbf{n}) = \int_{\mathcal{D}} ((|\nabla \mathbf{n}_2|^2 - |\nabla n_1|^2 - |\nabla n_2|^2) - \frac{1}{\xi^2} (\mathbf{n}_2^2 - n_2^2)) dx_1 dx_2$$

Now

$$\nabla \mathbf{n}_2 = \frac{n_1 \nabla n_1 + n_2 \nabla n_2}{n_2}$$

in $\mathcal{D}$.

Thus

$$|\nabla \mathbf{n}_2|^2 - |\nabla n_1|^2 - |\nabla n_2|^2 = \frac{|n_1 \nabla n_1 + n_2 \nabla n_2|^2}{n_2} - |\nabla n_1|^2 - |\nabla n_2|^2$$

$$= \frac{|n_1 \nabla n_1 + n_2 \nabla n_2|^2 - (|\nabla n_1|^2 + |\nabla n_2|^2)(n_1^2 + n_2^2)}{n_1^2 + n_2^2}$$

$$= -\frac{|n_1 \nabla n_2 - n_2 \nabla n_1|^2}{n_1^2 + n_2^2}.$$ 

It follows that

$$\mathbb{E}(\mathbf{n}) - \mathbb{E}(\mathbf{n}) = -\int_{\mathcal{D}} \left( \frac{|n_1 \nabla n_2 - n_2 \nabla n_1|^2}{n_1^2 + n_2^2} + \frac{1}{\xi^2} n_1^2 \right) dx_1 dx_2 < 0.$$ 

This implies that $\mathbf{n}$ is not a minimizer for $\mathbb{E}$, which is a contradiction.

Lemma 2.2 and Theorem 2.3 combine to give the following corollary.

**Corollary 2.4** Let $\mathbf{n} = (n_1, n_2, n_3)$ be a minimizer of $\mathbb{E}$ in $\mathcal{F}$ with nonnegative components. Then either $\mathbf{n} = (0, 1, 0)$, $\mathbf{n} = (0, 0, 1)$ or $\mathbf{n} = (0, n_2, n_3)$ with $0 < n_2, n_3 < 1$ in $\mathcal{D}$. 

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Proof: From the theorem we have \( n_1 \equiv 0 \). If \( n_2(x) = 0 \) for some \( x \in \mathcal{D} \) then the lemma implies that \( n_2 \equiv 0 \). Since \( n_2^2 + n_3^2 = 1 \) we see that \( n = (0,0,1) \). Similarly if \( n_3(x) = 0 \) for some \( x \in \mathcal{D} \) it follows that \( n = (0,1,0) \). The remaining possibility is \( 0 < n_2, n_3 < 1 \) in \( \mathcal{D} \).

The corollary allows us to introduce a new scalar variable, \( \theta \), such that \( 0 \leq \theta(x_1,x_2) \leq \pi/2 \) and for which

\[
(2.1) \quad n(x_1,x_2) = (0, \sin \theta, \cos \theta)
\]

Note \( \theta(x_1,x_2) \) is the angle between \( e_3 \) and \( n(x_1,x_2) \).

We see that \( \theta \) is analytic in \( \mathcal{D} \), \( \theta \in H^1(\mathcal{D}) \), and that

\[
(2.2) \quad E(n) = E(\theta)
\]

where

\[
E(\theta) = \int_{\mathcal{D}} (|\nabla \theta|^2 - \frac{1}{\zeta^2} \sin^2 \theta)dx_1dx_2 + \frac{\epsilon}{\nu} \int_S \sin^2 \theta d\mathcal{H}^1 + \frac{\beta}{\nu} \int_{S_e} \sin^2 \theta d\mathcal{H}^1.
\]

Moreover \( \theta \) is a solution to the following boundary value problem in the sense of \( H^1 \)

\[
 \Delta \theta + \left( \frac{1}{\zeta^2} \right) \sin \theta \cos \theta = 0 \quad \text{in} \ \mathcal{D},
\]

\[
(2.3) \quad \frac{\partial \theta}{\partial \nu} + \left( \frac{\epsilon}{\nu} \right) \sin \theta \cos \theta = 0 \quad \text{on} \ S,
\]

\[
\frac{\partial \theta}{\partial \nu} + \left( \frac{\beta}{\nu} \right) \sin \theta \cos \theta = 0 \quad \text{on} \ S_e,
\]

where \( \nu \) is the exterior normal to \( \mathcal{D} \).

Theorem 2.5 A function \( \theta \) is a minimizer for \( E \) in \( H^1(\mathcal{D}) \) if and only if there is an \( n \in \mathcal{F} \) minimizing \( E \) where \( \theta \) and \( n \) are related by (2.2). Moreover every minimizer \( \theta \) satisfies

\[
\frac{k\pi}{2} \leq \theta \leq \left( \frac{k+1}{2} \right) \pi \quad \text{for some integer} \ k
\]

and each bounded equilibrium \( \theta \in C^{2,\alpha}(\overline{\mathcal{D}} \setminus (\overline{\mathcal{S}} \cap \overline{\mathcal{S}_e})) \).
Remark. $\mathcal{S} \cap \mathcal{S}_e$ is the subset of the cross section where the edges of the polymer fibers intersect the exterior surface of the liquid crystal. In particular, if the fibers are all located in the interior of the composite then $\mathcal{S} \cap \mathcal{S}_e = \emptyset$.

Proof: We only need to verify the regularity of $\theta$ near $\partial \mathcal{D} \setminus (\mathcal{S} \cap \mathcal{S}_e)$. First we recall bounded weak solutions to (2.3) are of class $C^\gamma(\mathcal{D} \setminus (\mathcal{S} \cap \mathcal{S}_e))$ for some $\gamma > 0$ (see [14]). Second (using linear elliptic theory), weak solutions for which $\Delta \theta \in C^\gamma(\mathcal{D} \setminus (\mathcal{S} \cap \mathcal{S}_e))$ and for which $\frac{\partial \theta}{\partial \nu} \in C^\gamma(\partial \mathcal{D} \setminus (\mathcal{S} \cap \mathcal{S}_e))$ are such that $\theta \in C^{1,\gamma}(\mathcal{D} \setminus (\mathcal{S} \cap \mathcal{S}_e))$. (See [1].)Appealing to linear theory once more we have $\theta \in C^{2,\alpha}(\mathcal{D} \setminus (\mathcal{S} \cap \mathcal{S}_e))$.

Since we are assuming $n$ has nonnegative components we restrict our attention.

Corollary 2.6 If $\theta$ is a solution to (2.3) such that $0 \leq \theta \leq \frac{\pi}{2}$ then either $\theta \equiv 0$, $\frac{\pi}{2}$ or $0 < \theta < \frac{\pi}{2}$ for $x \in \mathcal{D} \setminus (\mathcal{S} \cap \mathcal{S}_e)$.

Proof: By the previous theorem $\theta$ is a classical solution to (2.3). The equation in (2.3) can be written as $\Delta \theta + \lambda(x)\theta = 0$. By the maximum principle if $\theta \geq 0$ then either $\theta > 0$ in $\mathcal{D}$ or $\theta \equiv 0$ in $\mathcal{D}$. At the boundary we see from (2.3) that $\frac{\partial \theta}{\partial \nu} = 0$ if $\theta = 0$ at some point in $\partial \mathcal{D} \setminus (\mathcal{S} \cap \mathcal{S}_e)$, contradicting the Hopf maximum principle. We conclude that either $\theta > 0$ in $\mathcal{D} \setminus (\mathcal{S} \cap \mathcal{S}_e)$ or $\theta \equiv 0$ in $\mathcal{D}$. Setting $\varphi := \frac{\pi}{2} - \theta$ an identical analysis for $\varphi$ implies either $\theta < \frac{\pi}{2}$ in $\mathcal{D} \setminus (\mathcal{S} \cap \mathcal{S}_e)$ or $\theta \equiv \frac{\pi}{2}$ in $\mathcal{D}$.

The normalized extrapolation length $b' = \frac{b}{w}$ indicates locally the strength of the bulk energy with respect to the surface energy of the polymer fibers. If the surface energy is relatively strong then $b'$ is small. In this paper we are interested in the case of small extrapolation length. More specifically we consider $b' = \epsilon^2$ where $\epsilon^{-1}$ is comparable to the total surface area of the polymer network $\mathcal{H}^1(\mathcal{S})$. This will allow solutions to develop order one oscillations on an $\epsilon$-lengthscale.

Remark. From this point forward we assume that $\beta$ is fixed and $\frac{K_{wW}}{wW} = \frac{b}{w} = b' = \epsilon^2$. Thus we have $M = W \chi H^2/w = \epsilon^2/\xi^2$. Setting $\mathcal{E}(\theta) := \epsilon^2 \mathcal{E}(\theta)$ we arrive at the energy (1.3) and the associated equilibrium problem (1.4).
3 Analysis of equilibriums for $\mathcal{E}$ in $H^1(D)$

In this section we assume that the parameter $\epsilon$ and the domain $D$ are fixed with $\epsilon > 0$. Moreover we assume that the polymer fibers are located in the interior of the composite. As such we assume that

$$\partial D = \mathcal{S} \cup \mathcal{S}_e$$

with $\mathcal{S} \cap \mathcal{S}_e = \emptyset$.

We will find two transition thresholds for $M$, $\mu$ and $\bar{\mu}$, such that if $0 \leq M \leq \mu$ then the director field (2.1) with minimal energy has $\theta \equiv 0$. A heterogeneous intermediate state is the minimizer if $\mu < M < \bar{\mu}$ and if $\mu \leq M$ the minimal state has $\theta \equiv \pi/2$.

We first prove a comparison lemma relating equilibrium configurations to field strengths.

**Lemma 3.1** Assume $\mathcal{S} \cap \mathcal{S}_e = \emptyset$. Consider solutions to (1.4) with $\theta = \theta_i$, $M = M_i$ for $i = 1, 2$ and $0 < \theta_1 < \pi/2$ in $\overline{D}$. If $M_1 < M_2$ then $\theta_1 < \theta_2$ in $\overline{D}$. If $M_1 = M_2$ then $\theta_1 = \theta_2$.

**Proof:** Let $F(\theta)$ be such that $F'(\theta) = 1/\sin \theta \cos \theta$. Set $v_1 = F(\theta_1)$ and $v_2 = F(\theta_2)$. Note that since $0 < \theta_1, \theta_2 < \pi/2$ then $-\infty < v_1, v_2 < \infty$ in $\overline{D}$, and that since $F'' > 0$ we have $\theta_1 < \theta_2$ if and only if $v_1 < v_2$. From the proof of Theorem 2.5 we have $v_1, v_2 \in C^2(\overline{D})$. From (1.4) then they each satisfy

$$\Delta v_i - \frac{F''(\theta_i)}{F'(\theta_i)^2} |\nabla v_i|^2 = -\frac{M_i}{\epsilon^2}$$

in $D$,

$$\frac{\partial v_i}{\partial \nu} = -\frac{1}{\epsilon} \quad \text{on } \mathcal{S},$$

$$\frac{\partial v_i}{\partial \nu} = -\frac{\beta}{\epsilon^2} \quad \text{on } \mathcal{S}_e$$

where $\theta_i = \theta(v_i) = F^{-1}(v_i)$. Set

$$g(v) = -\frac{F''(\theta(v))}{F'(\theta(v))^2}$$

$$= \cos(2\theta(v)).$$

Note

$$g'(v) = -\sin^2(2\theta(v)) < 0.$$
Thus $v_1$ and $v_2$ each satisfy
\[
\Delta v_i + g(v_i) |\nabla v_i|^2 = -\frac{M_i}{\epsilon^2} \text{ in } D, \\
(3.2)
\frac{\partial v_i}{\partial \nu} = -\frac{1}{\epsilon} \text{ on } S, \\
\frac{\partial v_i}{\partial \nu} = -\frac{\beta}{\epsilon^2} \text{ on } S_{\epsilon}.
\]
Set $z = v_1 - v_2$ and take the differences of the equations for $i = 1$ and $2$. This can be viewed as
\[
\Delta z + a \cdot \nabla z + dz = \frac{(M_2 - M_1)}{\epsilon^2} \text{ in } D, \\
\frac{\partial z}{\partial \nu} = 0 \text{ on } \partial D
\]
where $a = g(v_2)\nabla(v_1 + v_2)$ and
\[
d = |\nabla v_1|^2 \frac{(g(v_1) - g(v_2))}{(v_1 - v_2)}.
\]
Note from (3.1) that $d \leq 0$.

We apply the strong maximum principle to $z$. If $M_1 < M_2$ the principle asserts that either $z < 0$ in $\overline{D}$ or $z$ equals a nonnegative constant. Thus either
\[
(3.3) \quad v_1 < v_2 \text{ on } \overline{D} \quad \text{ or } \quad v_1 - v_2 \equiv \text{ Const.} \geq 0 \text{ in } \overline{D}.
\]
In the latter case we get
\[
(g(v_1) - g(v_2)) |\nabla v_1|^2 = (M_2 - M_1)/\epsilon^2 > 0 \text{ in } \overline{D}.
\]
In view of (3.1) this would be impossible.

If $M_1 = M_2$ we again conclude that (3.3) holds. The second case implies that
\[
(g(v_1) - g(v_2)) |\nabla v_1|^2 = 0 \text{ in } \overline{D}.
\]
Thus either $v_1 = v_2$ in $\overline{D}$ or $\nabla v_1 = 0$ in $\overline{D}$. Clearly the boundary conditions in (3.2) prohibit a constant solution. Thus either $v_1 < v_2$ in $\overline{D}$ or $v_1 = v_2$ in $\overline{D}$. Reversing the roles of $v_1$ and $v_2$ we see that $v_1 = v_2$.

In order to distinguish stable equilibrium we consider the second variation of $\mathcal{E}$. Let
\[
D^2 \mathcal{E}(\theta; w) = \left. \frac{d^2}{dt^2} \mathcal{E}(\theta + tw) \right|_{t=0} \text{ for } \theta, w \in H^1(D).
\]
Define $\mu$ and $\overline{\mu}$ such that
\[
\mu := \inf_{w \in H^1(D)} \left( \epsilon^2 \int_D |\nabla w|^2 \, dx_1 dx_2 + \epsilon \int_S w^2 d\mathcal{H}^1 + \beta \int_{S_\epsilon} w^2 d\mathcal{H}^1 \right)
\]
(3.4)
\[
= \inf_{w \in H^1(D)} \frac{1}{2} D^2 \mathcal{E}(0; w) + M
\]
and
\[
\overline{\mu} := -\inf_{w \in H^1(D)} \left( \epsilon^2 \int_D |\nabla w|^2 \, dx_1 dx_2 - \epsilon \int_S w^2 d\mathcal{H}^1 - \beta \int_{S_\epsilon} w^2 d\mathcal{H}^1 \right)
\]
(3.5)
\[
= M - \inf_{w \in H^1(D)} \frac{1}{2} D^2 \mathcal{E}(\pi/2; w)
\]
Note $\theta \equiv 0$ is a stable equilibrium if and only if $M < \mu$ and $\theta \equiv \pi/2$ is stable if and only if $M > \overline{\mu}$.

**Lemma 3.2** The critical values satisfy $\underline{\mu} < \overline{\mu}$.

**Proof:** Set $w = (\mathcal{H}^2(D))^{-1/2}$ in (3.4) and (3.5). Then
\[
\mu \leq (\epsilon \mathcal{H}^1(S) + \beta \mathcal{H}^1(S_\epsilon))/\mathcal{H}^2(D) \leq \overline{\mu}.
\]
Moreover equality occurs in either statement if and only if the constant function is an extremal for the corresponding eigenvalue problem. Constant functions do not satisfy the natural boundary conditions for extremal solutions thus both inequalities are strict.

The next theorem describes solutions to (1.4) for different values of $M$.

**Theorem 3.3** Assume $\mathcal{D}$ is such that $\overline{S} \cap S_\epsilon = \emptyset$. Consider solutions to (1.4) valued in $[0, \pi/2]$. If $0 \leq M \leq \mu$ or $\overline{\mu} \leq M$ then there are exactly two solutions, $\theta \equiv 0$ and $\theta \equiv \pi/2$. If $\underline{\mu} < M < \overline{\mu}$ there is exactly one more solution, $\theta(x, M)$. This solution satisfies
\[
0 < \theta(x, M) < \pi/2 \text{ for } x \in \overline{\mathcal{D}}.
\]
and is strictly increasing in $M$ for each $x$. 

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Extend $\theta(\cdot, M)$ by defining

$$\theta(\cdot, M) \equiv 0 \quad \text{if} \quad M \leq \mu,$$

$$\equiv \pi/2 \quad \text{if} \quad \mu \leq M.$$

Then for each $M \geq 0$, $\theta(\cdot, M)$ is the unique minimizer for $E$ among functions valued in $[0, \pi/2]$ moreover $\theta \in C([0, \infty); C^2(\overline{D}))$.

**Proof:** First consider $\mu < M < \overline{\mu}$. Then from (3.4) and (3.5) we have

$$\inf_{w \in H^1(\overline{D})} \frac{D^2E(0; w)}{\|w\|_2=1} = 2(\mu - M) < 0 \quad (3.6)$$

and

$$\inf_{w \in H^1(\overline{D})} \frac{D^2E(\pi/2; w)}{\|w\|_2=1} = 2(M - \overline{\mu}) < 0 \quad (3.7)$$

Thus neither $\theta \equiv 0$ or $\theta \equiv \pi/2$ are minimizers for $E$ for these values of $M$. On the other hand from Theorems 2.1 and 2.5 a minimizer exists valued in $[0, \pi/2]$. From Corollary 2.6 a third solution must take its values in $(0, \pi/2)$ and from Lemma 3.1 it must be unique. Thus $\theta(x, M)$ is uniquely determined for $\mu < M < \overline{\mu}$ as claimed. From Theorem 3.3, $\theta(x, M)$ is strictly increasing in $M$ for $\mu < M < \overline{\mu}$. By elliptic estimates and uniqueness we have

$$\theta(x, \cdot) \in C((\mu, \overline{\mu}); C^2(\overline{D})).$$

Next due to the monotonicity of $\theta(x, M)$ we see

$$\lim_{M \to \mu^+} \theta(x, M) =: \underline{w}(x) \quad \text{and} \quad \lim_{M \to \overline{\mu}^-} \theta(x, M) =: \overline{w}(x)$$

exist where the convergence is in $C^2(\overline{D})$. Moreover $0 \leq \underline{w} < \pi/2$ and $0 < \overline{w} \leq \pi/2$ in $\overline{D}$. We claim that $\underline{w} \equiv 0$ and $\overline{w} \equiv \pi/2$. To prove this for $\overline{w}$ we assume that it is not the case. Then by Corollary 2.6 we see that $0 < \underline{w} < \pi/2$ in $\overline{D}$. Since $\theta(\cdot, M)$ is minimizer if $\mu < M < \overline{\mu}$ it follows that the limit $\overline{w}$ is also a minimizer if $M = \mu$. Thus $E(\overline{w}) \leq E(0) = 0$.

Set $z = \sin(\underline{w}) \neq 0$. Then

$$\int_{\overline{D}} |\nabla z|^2 \mathrm{d}x_1 \mathrm{d}x_2 < \int_{\overline{D}} |\nabla \underline{w}|^2 \mathrm{d}x_1 \mathrm{d}x_2.$$
Thus
\[
\int_D (\epsilon^2 |\nabla z|^2 - \mu z^2) dx_1 dx_2 + \epsilon \int_S z^2 d\mathcal{H} + \beta \int_{S_e} z^2 d\mathcal{H}
\]
\[
< \int_D (\epsilon^2 |\nabla w|^2 - \mu z^2) dx_1 dx_2 + \epsilon \int_S z^2 d\mathcal{H} + \beta \int_{S_e} z^2 d\mathcal{H}
\]
\[
= \mathcal{E}(w) \leq 0.
\]
This contradicts the definition of $\mu$ in (3.4) and establishes that $\underline{w} \equiv 0$. The argument for showing that $\bar{w} \equiv \pi/2$ is similar. Thus we have proved all of the assertions for $\underline{\mu} \leq M \leq \bar{\mu}$.

We next consider $0 \leq M \leq \mu$. Let us assume that there is a solution, $\tilde{\theta}$, to (1.4) with $0 < \tilde{\theta} < \pi/2$. Given $\eta > 0$ choose $\delta > 0$ so that $0 < \theta(x, \underline{\mu} + \delta) < \eta$ in $\overline{D}$. Then using Lemma 3.1 we see
\[
0 < \tilde{\theta}(x) < \theta(x, \underline{\mu} + \delta) < \eta \text{ in } \overline{D}.
\]
Since $\eta$ is arbitrary we see that such a $\tilde{\theta}$ cannot exist and that $\underline{\theta} \equiv 0$ and $\bar{\theta} \equiv \pi/2$ are the only solutions for $M \leq \underline{\mu}$. Finally from (3.7) we see that $\underline{\theta} \equiv \pi/2$ is unstable for $M < \underline{\mu}$. Thus $\underline{\theta} \equiv \pi/2$ cannot be a minimizer for $\mathcal{E}$ if $M \leq \underline{\mu}$. It follows that $\theta \equiv 0$ is the unique minimizer in this case. The argument for $M \geq \bar{\mu}$ is identical.

\section{Periodic Networks}

\subsection{Fundamental Cells.}

By assuming more structure on the polymer network away from the boundary of the composite we are able to obtain further details concerning the transition fields and configurations. In order to describe these networks we denote
\[
Y := \{(x_1, x_2) | -\frac{1}{2} < x_1, x_2 < \frac{1}{2}\}
\]
as the fundamental unit cell,
\[
(4.1)
\]
representing the cross section of a fundamental polymer rod where $T$ is a simply connected open set with a $C^{2,\alpha}$ boundary. Define $Y^* := Y \setminus T$ as the region in $Y$ occupied by the liquid crystal. We set
\[
(4.2) \quad Y(\epsilon) := \{\epsilon x \mid x \in Y\}, \quad T(\epsilon) := \{\epsilon x \mid x \in T\} \text{ and } Y^*(\epsilon) := \{\epsilon x \mid x \in Y^*\}.
\]
For each $\epsilon > 0$ we tile $\mathbb{R}^2$ with translates of the $\epsilon$-cell $Y(\epsilon)$. Set

$$Y(\epsilon) := \mathbb{R}^2 \setminus \bigcup_{z \in Z \times Z} (T(e) + \epsilon z)$$

where $Z = \{ \ldots, -1, 0, 1, \ldots \}$. 

### 4.2 The Cell Problem

In this section we introduce an $\epsilon$-periodic variational problem defined on $Y(\epsilon)$ and collect the characteristic features of its solution. These characteristics will correspond to the feature found for solutions of (1.4) such that $\mathcal{S} \cap \overline{\mathcal{S}_e} = \emptyset$ and their proofs follow in the same manner. Let

$$\mathcal{P}(Y^*) = \{ w \in H^1(Y^*) :$$

$$w(-\frac{1}{2}, x_2) = w(\frac{1}{2}, x_2) \text{ for } -\frac{1}{2} \leq x_2 \leq \frac{1}{2} \text{ and}$$

$$w(x_1, -\frac{1}{2}) = w(x_1, \frac{1}{2}) \text{ for } -\frac{1}{2} \leq x_1 \leq \frac{1}{2} \}$$

Set

$$\mathcal{E}_p(w) = \int_{\mathcal{Y}^*} (|\nabla w|^2 - M \sin^2(w))dx_1dx_2 + \int_{\partial T} \sin^2(w)d\mathcal{H}^1.$$  \hspace{1cm} (4.3)

**Theorem 4.1** There exist minimizers for $\mathcal{E}_p$ in $\mathcal{P}(Y^*)$ taking values in $[0, \pi/2]$. 

**Theorem 4.2** Let $\varphi$ be a bounded equilibrium for $\mathcal{E}_p$ in $\mathcal{P}$ and $\tilde{\varphi}$ its 1-periodic extension to $Y(1)$. Then $\tilde{\varphi} \in C^{2,\alpha}(\overline{Y(1)})$ and it satisfies

$$\Delta \tilde{\varphi} + M \sin \varphi \cos \varphi = 0 \text{ in } Y(1),$$

$$\frac{\partial \tilde{\varphi}}{\partial \nu} + \sin \varphi \cos \varphi = 0 \text{ on } \partial Y(1).$$  \hspace{1cm} (4.4)

**Theorem 4.3** Let $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ be solutions to (4.3) with $0 < \tilde{\varphi}_1$, $\tilde{\varphi}_2 < \pi/2$ and where $0 \leq M_1 \leq M_2$. If $M_1 < M_2$ then $\tilde{\varphi}_1 < \tilde{\varphi}_2$ in $\overline{Y(1)}$. If $M_1 = M_2$ then $\tilde{\varphi}_1 = \tilde{\varphi}_2$.

Define

$$\lambda = \inf_{\|w\|_2 = 1} \left( \int_{Y^*} |\nabla w|^2 dx_1dx_2 + \int_{\partial T} w^2 d\mathcal{H}^1 \right).$$  \hspace{1cm} (4.5)

$$\bar{\lambda} = -\inf_{\|w\|_2 = 1} \left( \int_{Y^*} |\nabla w|^2 dx_1dx_2 - \int_{\partial T} w^2 d\mathcal{H}^1 \right).$$  \hspace{1cm} (4.6)
**Theorem 4.4** The infima are achieved in (4.5) and (4.6) by \( w, \tilde{w} \in C^{2,\alpha}(\overline{Y^*}) \) respectively such that \( 0 < w, \tilde{w} \) in \( Y^* \). Moreover the 1-periodic extensions of \( w \) and \( \tilde{w} \) are in \( C^{2,\alpha}(\overline{Y(1)}) \) and \( 0 < \underline{\lambda} < \overline{\lambda} \).

**Theorem 4.5** There exists \( \varphi \in C([0, \infty); C^{2,\alpha}(\overline{Y^*})) \) such that

1. for each \( M \geq 0 \), \( \varphi(\cdot, M) \) is the unique minimizer for \( E_p \) in \( P(\overline{Y^*}) \) for which \( 0 \leq \varphi \leq \pi/2 \).
2. \( \varphi(x, M) \equiv 0 \) if \( M \leq \underline{\lambda} \) and \( \varphi(x, M) \equiv \pi/2 \) if \( \overline{\lambda} \leq M \),
3. \( \varphi(x, M) \) is a strictly increasing in \( M \) for \( \underline{\lambda} \leq M \leq \overline{\lambda} \), for each \( x \in \overline{Y^*} \).

**4.3 Asymptotic Analysis of Critical Values**

In this section we consider a family of polymer networks determined by \( T = T(\epsilon) \) for \( \epsilon > 0 \) and corresponding solutions to (1.4). For each \( \epsilon > 0 \), \( T(\epsilon) \) is a finite union of disjoint open sets with regular boundaries as described in Section 2. We investigate \( \mu(\epsilon) \) and \( \overline{\mu}(\epsilon) \) as \( \epsilon \to 0 \). These quantities are defined by (3.4) and (3.5), the critical values at which the solutions \( \theta \equiv 0 \) and \( \theta \equiv \pi/2 \) respectively lose stability. Let \( D(\epsilon) \) be the domain of liquid crystal and \( S(\epsilon) \) be the interface.

For the analysis of \( \mu(\epsilon) \) we assume that \( \{T(\epsilon)\} \) have a given \( \epsilon \)-periodic structure. In this instance we show that \( \mu(\epsilon) \to \overline{\lambda} \) as \( \epsilon \to 0 \) where \( \overline{\lambda} \) is the lower critical value for the cell problem defined in Section 4.2.

**Theorem 4.6** Let \( T \) be the cross section for a fundamental fiber as in (4.1). Suppose that \( T(\epsilon) = \Omega \cap \bigcup_{z \in \mathbb{Z} \times \mathbb{Z}} (T(\epsilon) + \epsilon z) \). Then \( \lim_{\epsilon \to 0} \mu(\epsilon) = \overline{\lambda} \).

**Proof:** We derive an upper and a lower estimate for \( \mu(\epsilon) \).

**Step 1.** We construct a test function for (3.4) that vanishes near \( S_\epsilon \). Consider \( w(x) \), the positive extremal to (4.5) defined for \( x \in \overline{Y^*} \). Extend \( w \) as a periodic function to \( \overline{Y(1)} \). It follows from Theorem 4.4 that there are constants \( 0 < c_1 < c_2 \) so that

\[
(4.7) \quad c_1 \leq w \leq c_2,
\]

and

\[
(4.8) \quad w \in C^{2,\alpha}(\overline{Y(1)}).
\]

Define

\[
\underline{w}(x) = w(x/\epsilon) \quad \text{for} \quad x \in \overline{Y(\epsilon)}.
\]
Let $\phi_\epsilon \in C^1_c(\Omega)$ such that $0 \leq \phi_\epsilon \leq 1$, $\phi_\epsilon = 1$ if $x \in \Omega$ and $\text{dist}(x, \partial \Omega) \geq 2\epsilon$, $\phi_\epsilon = 0$ if $x \in \Omega$ and $\text{dist}(x, \partial \Omega) \leq \epsilon$, and $|\nabla \phi_\epsilon| \leq c_3/\epsilon$.

Set $\zeta_\epsilon := \phi_\epsilon w_\epsilon$ in $D(\epsilon)$. We insert $\zeta_\epsilon$ into (3.4). It follows that

$$
\mu(\epsilon) \leq \left( \epsilon^2 \int_{D(\epsilon)} |\nabla w_\epsilon|^2 \, dx_1 dx_2 + \epsilon \int_{S(\epsilon) \cap \partial M(\epsilon)} w_\epsilon^2 d\mathcal{H}^1 + c_4 \epsilon \right) / \int_{D(\epsilon)} \phi_\epsilon^2 w_\epsilon^2 \, dx_1 dx_2.
$$

Let $\mathcal{M}(\epsilon)$ be the union of the tiles, $Y^*(\epsilon) + \epsilon z$ for $z \in Z \times Z$ contained in $D(\epsilon)$. It follows that $D(\epsilon) \setminus \mathcal{M}(\epsilon)$ is contained in a $2\epsilon$ neighborhood of $S_\epsilon(\epsilon)$. Using (4.7) and (4.8) we see

$$
\mu(\epsilon) \leq \epsilon^2 \int_{\mathcal{M}(\epsilon)} |\nabla w_\epsilon|^2 \, dx_1 dx_2 + \epsilon \int_{S(\epsilon) \cap \partial \mathcal{M}(\epsilon)} w_\epsilon^2 d\mathcal{H}^1 + c_5 \epsilon.
$$

Let $N(\epsilon)$ be the number of tiles making up $\mathcal{M}(\epsilon)$. Then

$$
\frac{\epsilon^2 \int_{\mathcal{M}(\epsilon)} |\nabla w_\epsilon|^2 \, dx_1 dx_2 + \epsilon \int_{S(\epsilon) \cap \partial \mathcal{M}(\epsilon)} w_\epsilon^2 d\mathcal{H}^1}{\int_{\mathcal{M}(\epsilon)} w_\epsilon^2 dx_1 dx_2} = \frac{N(\epsilon) (\epsilon^2 \int_{Y^*(\epsilon)} |\nabla w_\epsilon|^2 \, dx_1 dx_2 + \epsilon \int_{T(\epsilon)} w_\epsilon^2 d\mathcal{H}^1)}{N(\epsilon) \int_{Y^*(\epsilon)} w_\epsilon^2 dx_1 dx_2} = \frac{(\int_{Y^*} |\nabla w_\epsilon|^2 \, dx_1 dx_2 + \int_{T} w_\epsilon^2 d\mathcal{H}^1)}{\int_{Y^*} w_\epsilon^2 dx_1 dx_2} = \lambda
$$

where we used (4.5).

Thus we see

$$(4.9) \quad \mu(\epsilon) \leq \lambda + c_5 \epsilon.
$$

_Step 2_. Let $u_\epsilon$ be the positive extremal to (3.4). Set $v_\epsilon := u_\epsilon / w_\epsilon$. Using (4.7) and (4.8) we have $v_\epsilon \in H^1(D(\epsilon))$. Thus inserting $u_\epsilon$ into (3.4) we have

$$
\mu(\epsilon) = \epsilon^2 \int_{D(\epsilon)} \nabla w_\epsilon \cdot \nabla (w_\epsilon v_\epsilon^2) \, dx_1 dx_2
$$

$$
+ \epsilon^2 \int_{D(\epsilon)} w_\epsilon^2 |\nabla v_\epsilon|^2 \, dx_1 dx_2
$$

$$
+ \epsilon \int_{S(\epsilon)} w_\epsilon^2 v_\epsilon^2 d\mathcal{H}^1 + \beta \int_{S(\epsilon)} w_\epsilon^2 v_\epsilon^2 d\mathcal{H}^1.
$$
Integrating by parts in the first term on the right and using the fact that \( w_\epsilon \) satisfies
\[
\epsilon^2 \Delta w_\epsilon = -\lambda w_\epsilon \text{ in } \mathcal{Y}(\epsilon),
\]
\[
\epsilon \frac{\partial w_\epsilon}{\partial \nu} = -w_\epsilon \text{ on } \partial \mathcal{Y}(\epsilon)
\]
we get
\[
\mu(\epsilon) \geq \lambda + \int_{S_\epsilon(\epsilon)} (\epsilon^2 \frac{\partial w_\epsilon}{\partial \nu} + \beta w_\epsilon) w_\epsilon v_\epsilon^2 dH^1.
\]
Since \( |\nabla w_\epsilon| \leq c_3/\epsilon \) and \( w_\epsilon \geq c_1 > 0 \) we see \((\epsilon^2 \frac{\partial w_\epsilon}{\partial \nu} + \beta w_\epsilon) > 0\) for \( \epsilon \) sufficiently small. Thus \( \mu(\epsilon) \geq \lambda \) for \( \epsilon \) small enough.

Combining this inequality with (4.9) proves the theorem.

We now show that as long as \( S_\epsilon(\epsilon) \) is at least a fixed portion of the exterior boundary of \( D(\epsilon) \) then \( \overline{\mu}(\epsilon) \rightarrow \infty \) as \( \epsilon \rightarrow 0 \). This is independent of the nature of \( T(\epsilon) \) inside the composite and is markedly different from what occurs for the cell problem where the upper transition threshold, \( \overline{\lambda} \), is finite.

**Theorem 4.7** Assume there is a \( \delta > 0 \) so that \( H^1(S_\epsilon(\epsilon)) \geq \delta \) for all \( \epsilon \) sufficiently small. Then
\[
\lim_{\epsilon \to 0} \epsilon \cdot \overline{\mu}(\epsilon) > 0.
\]

**Proof:** Let \( \eta_\epsilon = 1 - \phi_\epsilon \) where \( \phi_\epsilon \) is the cut-off function defined in the previous theorem. Inserting \( \eta_\epsilon \) as a test function in (3.5) we see
\[
\overline{\mu}(\epsilon) \int_\Omega \eta_\epsilon^2 dx_1 dx_2 \geq -\epsilon^2 \int_\Omega |\nabla \eta_\epsilon|^2 dx_1 dx_2 + \beta \int_{S_\epsilon(\epsilon)} \eta_\epsilon^2 dH^1.
\]
Thus \( \epsilon \overline{\mu}(\epsilon) \geq c_1(\beta \delta - c_2 \epsilon) \) for some constants \( c_1, c_2 > 0 \), for all \( \epsilon > 0 \) and sufficiently small.

### 4.4 Asymptotic Analysis of minimizers

In the previous section we saw that the critical fields are sensitive to the structure of \( T(\epsilon) \) near the exterior boundary \( S_\epsilon \) even if \( T(\epsilon) \) is assumed to be \( \epsilon \)-periodic. In this section we show that if the \( T(\epsilon) \) are \( \epsilon \)-periodic in the bulk of the composite, away
from $S_\epsilon$, then the solutions to the cell problem found in Section 4.2 give an accurate
description of minimizers for (1.3) away from $S_\epsilon$.

Given a fundamental cross section $T = T(1)$, as in (4.1), we introduce a bulk energy
defined on open bounded subsets of $\mathcal{Y}(\epsilon)$. Fix $\epsilon > 0$ and $M \geq 0$.

Let $\mathcal{O}$ be a bounded open subset of $\mathcal{Y}(\epsilon)$ with a Lipschitz continuous boundary. Set

$$B(w; \epsilon, M, \mathcal{O}) = \int_\mathcal{O} (\epsilon^2 |\nabla w|^2 - M \sin^2 w) \, dx_1 dx_2 + \epsilon \int_{\mathcal{O} \cap \partial \mathcal{Y}(\epsilon)} \sin^2 w \, d\mathcal{H}^1$$

for $w \in H^1(\mathcal{O})$.

**Definition.** For $\epsilon$, $M$, and $\mathcal{O}$ fixed we say that $w \in H^1(\mathcal{O})$ is a local minimizer
for $B$ if

$$B(w; \epsilon, M, \mathcal{O}) \leq B(v; \epsilon, M, \mathcal{O}) \text{ for all } v \in H^1(\mathcal{O})$$

such that $w = v$ on $\partial \mathcal{O} \setminus \partial \mathcal{Y}(\epsilon)$.

For $n \in \mathbb{N}$ and $\epsilon > 0$. Let $\mathcal{C}(n, \epsilon)$ be the perforated cell

$$\mathcal{C}(n, \epsilon) = \{(x_1, x_2) : (-n + \frac{1}{2})\epsilon < x_1, x_2 < (n - \frac{1}{2})\epsilon\} \cap \mathcal{Y}(\epsilon).$$

Note that $\mathcal{C}(n, \epsilon)$ is tiled by $(2n - 1)^2$ translations of $\overline{Y^*}(\epsilon)$.

We first prove several estimates for local minimizers. Denote by $\partial_e \mathcal{C}(n, \epsilon)$ the ex-
terior boundary for $\mathcal{C}(n, \epsilon)$,

$$\partial_e \mathcal{C}(n, \epsilon) = \{(x_1, x_2) : x_1 \text{ or } x_2 \in \{(-n + \frac{1}{2})\epsilon, (n - \frac{1}{2})\epsilon\}\}.$$

We distinguish several functions used for comparison and describe their properties. Consider

$$u \in C^2(\overline{\mathcal{C}(n, \epsilon)} \setminus \partial_e \mathcal{C}(n, \epsilon)) \cap C(\overline{\mathcal{C}(n, \epsilon)})$$

satisfying

$$\epsilon^2 \Delta u + M \sin u \cos u = 0 \text{ in } \mathcal{C}(n, \epsilon),$$

$$\frac{\partial u}{\partial \nu} + \sin u \cos u = 0 \text{ on } \partial \mathcal{C}(n, \epsilon) \setminus \partial_e \mathcal{C}(n, \epsilon),$$

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(4.12) \[ u = u^0 \text{ on } \partial_e C(n, \epsilon). \]

If \( u^0 \in H^{1/2}(\partial_e C(n, \epsilon)) \cap C(\partial_e C(n, \epsilon)) \) and \( 0 \leq u^0 \leq \pi/2 \) the methods from Section 2 can be applied to show that a solution, \( u \), to (4.10)-(4.12) exists and that \( 0 \leq u \leq \pi/2 \). Moreover if \( 0 < u^0 < \pi/2 \) then the solution is unique, \( 0 < u < \pi/2 \) in \( C(\partial_e C(n, \epsilon)) \) and using the proof of Lemma 3.1 we can compare solutions. Specifically if \( \tilde{u} \) solves (4.10), (4.11) with \( M \) replaced by \( \tilde{M} \) such that \( \tilde{M} \leq M \), assuming \( 0 < u \), and \( 0 \leq \tilde{u} \leq u < \pi/2 \) on \( \partial_e C(n, \epsilon) \) then it follows that \( \tilde{u} \leq u \) in \( C(n, \epsilon) \).

To go further, let \( u_a \) solve (4.10)-(4.12) with \( u^0 \equiv a \) where \( a \) is a constant such that \( 0 < a < \pi/2 \). Then \( u \) is increasing in \( a \). Set \( \overline{u}(x, \epsilon, M, n) := \lim_{a \uparrow \pi/2} u_a(x, \epsilon, M, n) \). Here the convergence is in \( C(\overline{C(n, \epsilon)}) \cap H^1(C(n, \epsilon)) \). We see the function \( \overline{u}(x, \epsilon, M, n) \) has the properties

\[ \overline{u}(x, \epsilon, \tilde{M}, n) \leq \overline{u}(x, \epsilon, M, n) \text{ if } \tilde{M} < M \]

and

\[ u(x) \leq \overline{u}(x, \epsilon, M, n) \]

for any solution, \( u \), to (4.10)-(4.12) with \( 0 \leq u^0 < \pi/2 \).

Next we note (arguing just as in Section 2) that there is a local minimizer, \( w \), for \( B(\cdot; \epsilon, M, C(n, \epsilon)) \) with \( w = a \) on \( \partial_e C(n, \epsilon) \) taking values in \( (0, \pi/2) \). Since \( u_a \) is the unique solution to (4.10)-(4.12) with values in \( (0, \pi/2) \) and \( u^0 = a \) it must be a local minimizer. As the limit of local minimizers is a local minimizer we have that \( \overline{u} \) is a local minimizer with \( u^0 = \pi/2 \).

Similarly we can construct the local minimizer \( \underline{u}(x, \epsilon, M, n) := \lim_{a \downarrow 0} u_a(x, \epsilon, M, n) \) having the properties

\[ \underline{u}(x, \epsilon, \tilde{M}, n) \leq \underline{u}(x, \epsilon, M, n) \text{ if } \tilde{M} < M \]

and

\[ \underline{u}(x, \epsilon, M, n) \leq u(x) \]

for any solution, \( u \), to (4.10)-(4.12) with \( 0 < u^0 \leq \pi/2 \).

We are now prepared to systematically derive estimates for local minimizers.
Lemma 4.8 Given constants $\bar{M}$ and $\underline{M}$ such that $\lambda < \underline{M} \leq M \leq \bar{M} < \lambda$ there is an integer $n_0$ and a constant $\rho > 0$ so that if $0 \leq M \leq \bar{M}$ then

$$\overline{u}(x, \epsilon, M, n_0) \leq \frac{\pi}{2} - \rho \text{ for } x \in C(1, \epsilon).$$

If $\bar{M} \leq M$ then we have

$$\rho \leq \underline{u}(x, \epsilon, M, n_0) \text{ for } x \in C(1, \epsilon).$$

Proof: We first rescale the problem by setting $\epsilon = 1$. Indeed if $w_\epsilon(x)$ is defined on $C(n, \epsilon)$ then by setting $\epsilon y = x$ for $y \in C(n, 1)$ and $w(y) := w_\epsilon(x)$ we see

$$\epsilon^2 B(w; 1, M, C(n, 1)) = B(w_\epsilon; \epsilon, M, C(n, \epsilon)).$$

Thus we can rescale without loss of generality.

We begin by finding an integer $n_1 > 4$ and a function $w \in H^1(C(n_1, 1))$ with $w = 0$ on $\partial C(n_1, 1)$ so that

$$B(w; 1, n_1, \rho < 0. \quad (4.13)$$

Let $\varphi = \varphi(x, M)$ be as in Theorem 4.5 and let $\tilde{\varphi}$ be its 1-periodic extension to $\overline{\mathbb{Y}}(1)$. Let $\zeta \in C^2((-n_1 + \frac{1}{2}, n_1 - \frac{1}{2}) \times (-n_1 + \frac{1}{2}, n_1 - \frac{1}{2}))$ be a cut off function where $0 \leq \zeta \leq 1$, $|\nabla \zeta | \leq 2$ and $\zeta = 1$ on $[-n_1 + \frac{3}{2}, n_1 - \frac{3}{2}] \times [-n_1 + \frac{3}{2}, n_1 - \frac{3}{2}]$. Set $w = \zeta \tilde{\varphi}$. Then there is a constant $A$ so that

$$B(w; 1, M, C(n_1, 1)) \leq B(\tilde{\varphi}; 1, M, C(n_1 - 1, 1)) + An_1 = (2n_1 - 3)^2 E_p(\varphi, M) + An_1.$$ 

Since $\lambda < \underline{M}$ we have $E_p(\varphi, M) < 0$ and (4.13) follows.

Thus $u \equiv 0$ is not a local minimizer on $C(n_1, 1)$. Now consider $\underline{u}(x, 1, M, n_1)$, the local minimizer constructed above. As a nonnegative solution to (4.10), (4.11) it must be that either $u \equiv 0$ or $u > 0$ in $C(n_1, 1) \setminus \partial C(n_1, 1)$. We have just shown that $u \not\equiv 0$. Thus

$$\inf_{x \in C(1, 1)} \underline{u}(x, 1, M, n_1) =: \rho_1(n_1) > 0.$$ 

Moreover since $\underline{u}(x, 1, M, n_1) \geq \underline{u}(x, 1, M, n_1)$ for $M > \bar{M}$ we have

$$\underline{u}(x, 1, M, n_1) \geq \rho_1(n_1) \text{ on } C(1, 1) \text{ for all } M \geq \bar{M}.$$ 

In the same manner, by showing that $u \equiv \frac{\pi}{2}$ is not a local minimum on $C(n_2, 1)$ for $n_2$ sufficiently large,
It follows that
\[ u(x, 1, M, n_2) \leq \bar{u}(x, 1, M, n_2) \leq \sup_{x \in \mathcal{C}(1,1)} u(x, 1, M, n_2) = \frac{\pi}{2} - \rho_2(n_2) \text{ for } x \in \mathcal{C}(1,1) \text{ and } M \leq \overline{M} \]
where \( \rho_2(n_2) > 0. \)

The assertions follow by taking \( n_0 = \max(n_1, n_2) \) and \( \rho = \min(\rho_1, \rho_2). \) \( \blacksquare \)

Let \( \varphi_\epsilon(x, M) = \tilde{\varphi}(x/\epsilon, M) \) where \( \tilde{\varphi} \) is the 1-periodic solution to the cell problem (4.4).

**Lemma 4.9** Given \( \eta > 0 \) there is an integer \( n_1(\eta) \) so that for all \( \epsilon > 0 \) and \( M \geq 0 \) we have
\[(4.14)\]
\[ |u(x, \epsilon, M, n_1) - \tilde{\varphi}_\epsilon(x, M)| < \eta, \quad |\bar{u}(x, \epsilon, M, n_1) - \tilde{\varphi}_\epsilon(x, M)| < \eta \text{ for } x \in \overline{\mathcal{C}(1, \epsilon)}. \]

**Proof:** We rescale as before setting \( \epsilon = 1 \) and carry out the proof for \( u \). The argument for \( \bar{u} \) is similar.

Choose \( \lambda < M < \overline{M} < \tilde{\lambda} \) so that
\[(4.15)\]
\[ |\tilde{\varphi}(x, M)| < \eta/2 \quad \text{and} \quad |\tilde{\varphi}(x, \overline{M}) - \pi/2| < \eta/2 \text{ for all } x. \]

By Theorem 4.5 this is always possible.

Next we consider \( n \geq n_0 \) where \( n_0 \) is determined in Lemma 4.8 and \( M \) such that \( \underline{M} \leq M \leq \overline{M} \). Now either
\[ u(x, 1, M, n) \equiv 0 \]
or
\[ \frac{\pi}{2} > u(x, 1, M, n) > 0 \text{ on } \overline{\mathcal{C}(n, 1)} \setminus \partial \mathcal{C}(n, 1). \]

Since \( u \) is a local minimizer the former is ruled out by the proof of Lemma 4.8. We now compare \( u(x, 1, M, n) \) with \( u(x - z, 1, M, n_0) \) and \( \bar{u}(x - z, 1, M, n_0) \) for \( x \in \mathcal{C}(n_0, 1) + z \), for those \( z \in Z \times Z \) such that \( \mathcal{C}(n_0, 1) + z \subset \mathcal{C}(n, 1) \). We get
\[ u(x - z, 1, M, n_0) \leq u(x, 1, M, n) \leq \bar{u}(x - z, 1, M, n_0) \]
for such \( u \), with \( x \in \mathcal{C}(n_0, 1) + z \). From Lemma 4.8 then we see that for any compact \( \mathcal{K} \subset \mathbb{R}^2 \) and \( n \) large enough
\[ 0 < \rho \leq u(x, 1, M, n) \leq \frac{\pi}{2} - \rho \text{ for } x \in \mathcal{K} \cap \overline{\mathcal{Y}(1)} \]
provided \( M \leq M \leq \overline{M} \).

Using this and elliptic estimates we can extract a subsequence \( \{u(x, 1, M_i, n_i)\} \) such that \( n_i \to \infty \) and \( M_i \to \tilde{M} \) for some \( \tilde{M} \), with \( M \leq \tilde{M} \leq \overline{M} \) that converges in \( C^2(K \cap \overline{Y(1)}) \), for each compact set \( K \) to a solution, \( \tilde{u} \), of

\[
\Delta \tilde{u} + \tilde{M} \sin \tilde{u} \cos \tilde{u} = 0 \quad \text{in} \quad Y(1),
\]

(4.16)

\[
\frac{\partial \tilde{u}}{\partial \nu} + \sin \tilde{u} \cos \tilde{u} = 0 \quad \text{on} \quad \partial Y(1),
\]

(4.17)

and such that

\[
0 < \rho < \tilde{u} < \frac{\pi}{2} - \rho \quad \text{on} \quad \overline{Y(1)}.
\]

We now show that \( \tilde{u} = \tilde{\varphi}(x, \tilde{M}) \). Note \( \tilde{\varphi} \) satisfies (4.16) and (4.17) and is uniformly bounded away from 0 and \( \frac{\pi}{2} \) as well.

We argue by contradiction. Assume for definiteness that \( \tilde{u}(x_0) - \tilde{\varphi}(x_0, \tilde{M}) > 0 \) for some \( x_0 \in \overline{Y(1)} \). Set \( v_1 = F(\tilde{u}) \) and \( v_2 = F(\tilde{\varphi}) \) where \( F \) is defined in Lemma 3.1. Note \( v_1 \) and \( v_2 \) are uniformly bounded in \( \overline{Y(1)} \) and \( v_1(x_0) - v_2(x_0) > 0 \). The argument in the proof of Lemma 3.1 asserts that \( v_1 - v_2 \) cannot attain a positive maximum in \( \overline{Y(1)} \).

It follows that there must be a sequence \( \{x_n\} \subset \overline{Y(1)} \) such that \( |x_n| \to \infty \) as \( n \to \infty \) and

\[
\lim_{n \to \infty} (v_1(x_n) - v_2(x_n)) = \sup_{\overline{Y(1)}} (v_1 - v_2) > 0.
\]

Choose \( z_n \in Z \times Z \) so that \( x_n \in \overline{C(1,1)} + z_n \). Set

\[
w_{1,n}(x) := v_1(x + z_n), \quad w_{2,n}(x) := v_2(x + z_n).
\]

There exist subsequences, \( w_{1,n_i} \) and \( w_{2,n_i} \), converging in \( C^2(K \cap \overline{Y(1)}) \) for each compact set \( K \) to \( \tilde{w}_1 \) and \( \tilde{w}_2 \) respectively where these limits satisfy (4.16) and (4.17). However \( \tilde{w}_1 - \tilde{w}_2 \) achieves a positive maximum at some point in \( \overline{C(1,1)} \) and this is a contradiction.

Thus \( \tilde{u} = \tilde{\varphi} \) on \( \overline{Y(1)} \). This implies that the full sequence, \( \{u(x, 1, M, n)\} \) converges uniformly to \( \tilde{\varphi}(x, M) \) on compact subsets of \( Y(1) \) for each \( M \leq M \leq \overline{M} \). Moreover the convergence is uniform in \( M \) for \( M \leq M \leq \overline{M} \) as well.

In particular given \( \eta > 0 \) there is an \( n_1(\eta) \geq n_0 \) so that with \( K = \overline{C(1,1)} \)

\[
|u(x, 1, M, n_1) - \tilde{\varphi}(x, M)| < \frac{\eta}{2} \quad \text{for} \quad x \in \overline{C(1,1)}
\]

(4.18)
provided $\underline{M} \leq M \leq \overline{M}$. Now if $M < \underline{M}$ then

\begin{equation}
0 \leq u(x, 1, M, n_1) \leq u(x, 1, \underline{M}, n_1) \leq |u(x, 1, M, n_1) - \tilde{\varphi}(x, M)| + \tilde{\varphi}(x, M) < \frac{\eta}{2} + \frac{\eta}{2} \quad \text{on } \overline{C(1,1)}.
\end{equation}

where we have used the monotonicity of $u$ in $M$, (4.15), and (4.18). Similarly if $M > \overline{M}$ we have

\[
0 \leq \frac{\pi}{2} - u(x, 1, M, n_1) \leq \frac{\pi}{2} - u(x, 1, \overline{M}, n_1) = \frac{\pi}{2} - \tilde{\varphi}(x, \overline{M}) + (\tilde{\varphi}(x, \overline{M}) - u(x, 1, \overline{M}, n_1)) < \frac{\eta}{2} + \frac{\eta}{2}.
\]

Combining this, (4.18) and (4.19) completes the proof of (4.14).

We next introduce a family of cross sections $\{T(\epsilon)\}$ that are eventually $\epsilon$-periodic on every compact subset of $\Omega$ as $\epsilon \to 0$.

**Definition.** We say that the family $\{T(\epsilon) : \epsilon > 0\}$ is locally periodic if there is a fundamental cross section $T$, as in (4.1), and a function $r(\epsilon) \geq 2\epsilon$ for $\epsilon > 0$ such that

\[
\lim_{\epsilon \to 0} r(\epsilon) = 0,
\]

for which

\[
T(\epsilon) \cap \{x \in \Omega : \text{dist}(x, \partial \Omega) > r(\epsilon)\} = \bigcup_{z \in \mathbb{Z} \times \mathbb{Z}} (T(\epsilon) + \epsilon z) \cap \{x \in \Omega : \text{dist}(x, \partial \Omega) > r(\epsilon)\} \quad \text{for each } \epsilon > 0.
\]

We can now state the main result for this section.

**Theorem 4.10** Let $\{T(\epsilon) : \epsilon > 0\}$ be a family that is locally periodic and let $\{\theta_\epsilon\}$ be a family of minimizers for (1.3) in $H^1(D(\epsilon))$ such that $0 \leq \theta_\epsilon \leq \frac{\pi}{2}$. Then if $K$ is a compact subset of $\Omega$ we have

\[
\lim_{\epsilon \to 0} \|\theta_\epsilon(x, M) - \tilde{\varphi}_\epsilon(x, M)\|_{C(K \cap D(\epsilon))} = 0.
\]

Moreover the convergence is uniform in $M$ for $M \geq 0$.

**Proof:** Let $K$ be a compact subset of $\Omega$. Recall $D(\epsilon) = \Omega \setminus T(\epsilon)$. Given $\eta > 0$, choose $\underline{\lambda} < \underline{M} < \overline{M} < \overline{\lambda}$ so that $\tilde{\varphi}(x, \underline{M}) < \eta/2$ and $\frac{\eta}{2} - \tilde{\varphi}(x, \overline{M}) < \eta/2$ for all $x$. Next set
n := max(n_0(M(\eta), \overline{M}(\eta)), n_1(\eta)) where n_0 was introduced in Lemma 4.8 and n_1 in Lemma 4.9. Finally we take \epsilon_0 sufficiently small so that for \epsilon < \epsilon_0 the family
\[ \mathcal{N} := \{ \overline{C(1, \epsilon)} + \epsilon z : \text{where } z \in Z \times Z \text{ such that } \overline{C(n, \epsilon)} + \epsilon z \subset \{ x \in \Omega : \text{dist}(x, \partial \Omega) > r(\epsilon) \} \}, \]
forms a cover for \( K \cap D(\epsilon) \). Thus we are requiring that the sets \( \overline{C(n, \epsilon)} + \epsilon z \) are contained in the region of \( D(\epsilon) \) for which \( T(\epsilon) \) is \( \epsilon \)-periodic.

Let \( M \leq M \leq \overline{M} \) and let \( \theta_\epsilon(x, M) \) be a minimizer to (1.3) in \( H^1(D(\epsilon)) \) with \( 0 \leq \theta_\epsilon \leq \frac{\pi}{2} \). According to Corollary 2.6 either \( \theta_\epsilon \equiv 0, \theta_\epsilon \equiv \frac{\pi}{2} \) or \( 0 < \theta_\epsilon < \frac{\pi}{2} \) in \( D(\epsilon) \) \( \setminus S_\epsilon \).

By the choice of \( \epsilon \) and \( n \), the set \( D(\epsilon) \) contains a cell, \( \overline{C(n, \epsilon)} + \epsilon z \) for some \( z \in Z \times Z \). Now \( \theta_\epsilon \) is a local minimizer on \( \overline{C(n, \epsilon)} + \epsilon z \) and the proof of Lemma 4.8 asserts that \( \theta_\epsilon \) can be neither 0 or \( \frac{\pi}{2} \). Thus \( 0 < \theta_\epsilon < \frac{\pi}{2} \) in \( D(\epsilon) \) \( \setminus S_\epsilon \).

Next for \( x' \in K \cap D(\epsilon) \) select an element in \( \mathcal{N} \) containing \( x', \overline{C(1, \epsilon)} + \epsilon z' \). We can compare \( u(x - \epsilon z', \epsilon, M, n), \theta_\epsilon(x, M) \) and \( v(x - \epsilon z', \epsilon, M, n) \) on \( \overline{C(n, \epsilon)} + \epsilon z' \) and apply Lemma 4.9. We obtain
\[ \tilde{\varphi}_\epsilon(x', M) - \eta < u \leq \theta_\epsilon(x', M) \leq \varphi(x', M) < \tilde{\varphi}_\epsilon(x', M) + \eta. \]

Thus we have
\[ | \theta_\epsilon(x, M) - \tilde{\varphi}_\epsilon(x, M) | < \eta \text{ for all } x \in K \cap D(\epsilon), \ M \leq M \leq \overline{M} \text{ and } \epsilon < \epsilon_0. \]

The cases of \( M < \overline{M} \) and \( \overline{M} \leq M \) are treated just as in the proof of Lemma 4.9. Thus the proof is complete.

**Remark.** Since \( \theta_\epsilon \) and \( \varphi_\epsilon \) are bounded we see that under the hypotheses of Theorem 4.10 that
\[ \lim_{\epsilon \to 0} \| \theta_\epsilon(x, M) - \tilde{\varphi}_\epsilon(x, M) \|_{L^2(D(\epsilon))} = 0. \]

### 5 Discussion

In this paper, using a fibril model with a fixed polymer network, we describe the equilibrium configurations for a polymer stabilized liquid crystal. In a two-dimensional setting, we show the existence of lower and upper critical values for an external magnetic field such that the minimizer is non constant when the field is between these two thresholds (i.e., the liquid crystal configuration is non-uniform). When the field is below the lower critical value, the configuration is uniform in the direction determined by the polymer network and boundary conditions. If the field is above the upper critical...
value, the configuration is uniform in the direction of the external magnetic field. We also study the properties of these configurations under various magnitudes of external fields.

In the case of a uniform polymer network with a periodic structure, as the number of polymer fibers goes to infinity, we show away from $S_{e}$ that minimizers become uniformly close to the oscillating solutions of the scaled cell problem. Moreover the lower critical values of the external field for the composite system approach those of the corresponding cell problem. Thus, as $\epsilon \to 0$, minimizing configurations are parallel to the easy axis, $e_{3}$, for nearly the same values of magnetic intensity as for the cell problem. However the situation is different for the upper critical values. The exterior boundary condition forces the upper critical values of the external field to tend to infinity as the number of fibers increases. This result also holds if the polymer network is not uniformly distributed. Thus as $\epsilon \to 0$, we have the phenomenon of minimizers being close but not identically parallel to the direction of the applied field, $h$, for $M = W_{\chi}H^{2}/w$ in a large interval to the right of the upper critical value for the cell problem, $\lambda$. This seems to be the most significant effect of the exterior boundary.

References


