Lab 13

MATLAB is a computational language, meaning that it is good for "number crunching". There are languages available, notably MAPLE and MATHEMATICA, which are symbolic languages. This means they can handle computations involving symbols.

MATLAB does have a symbolic toolbox (at least some installations do) which enables it to do some symbolic operations as well. In this lab we shall use this feature to study Laplace transforms.

1. In order to use the symbolic operations, we must define string expressions. This is done by enclosing the expression in single quotes. For example, if we create a string by \( y = 'x^2 + \log(x)' \) then we can differentiate and integrate \( y \) by \( \text{diff}(y) \) and \( \text{int}(y) \). We can also take higher derivatives. Try \( \text{diff}(y,2) \) for instance. Also, since this is all symbolic, we can carry along other variables. Suppose we take \( w \) as given by \( w = '(x+t^2) + \log(x+3*y)' \) Then we can differentiate with respect to \( x \) by \( \text{diff}(w,'x') \). Now, to Laplace transforms, we can compute the transforms by \( \text{laplace}(y) \) where \( y \) is a string defining the function, and inverse Laplace transform of a function \( F \) would be given by \( \text{invlaplace}(F) \).

Create strings for three functions of the type we have studied in the text, and compute their Laplace transforms. Then try the transforms of the derivatives by combining \( \text{laplace} \) with \( \text{diff} \). Then take the inverse transformations with \( \text{invlaplace} \). Record your observations. When doing this, use \( \exp(x) \) for \( e^x \). Also, before you use \( \text{invlaplace} \) use \( \text{laplace} \), otherwise (a MAPLE quirk) some of the routines don’t get loaded.

2. In our study of Laplace transforms, we have been considering only piecewise continuous functions of exponential type, that is functions \( y(t) \) which satisfy the condition

\[
|y(t)| < Ae^{Mt}
\]

for some constants \( A \) and \( M \). The reason for this is that this condition ensures that the integral defining the Laplace transform converges, at least for \( s \) larger than some number \( a \) (which can be taken to be the constant \( M \) in (1)). The proof of this is simply due to the fact that by (1), the area under the graph of \( |y(t)e^{-st}| \) is bounded above by the integral

\[
\int_0^\infty e^{Mt}e^{-st}dt,
\]

which converges to \( 1/(s-M) \) for \( s > M \). Write out the proof of this fact carefully, noting in the process that if \( F(s) \) is the Laplace transform of a function \( y(t) \) of exponential type, then \( F(s) \to 0 \) as \( s \to \infty \). In shorthand notation, we could say that “\( F(\infty) = 0 \)”. Now suppose we can differentiate under the integral sign in the formula for \( F(s) \) to obtain a formula for \( F'(s) \). This is actually valid by
a theorem in calculus called “Leibniz’s rule”. Derive the formula for $F'$, and in
the process, show that $F'(s) \to 0$ as $s \to \infty$, and the fact that

\begin{equation}
\mathcal{L}(t(y(t))) = -F'(s).
\end{equation}

3. Using \texttt{laplace}, compute the Laplace transform of

\[ y(t) = \frac{e^{-1/(4t)}}{\sqrt{t}}. \]

What is the result?

4. Next use \texttt{laplace} to compute the Laplace transform of

\[ u(t) = \sqrt{t}e^{-1/(4t)}. \]

Show by calculation (not computer) the same answer can also be derived from
formula 2 and Exercise 3. Show all your computations.

5. You might be getting the feeling that you are perhaps wasting your time learning
all this math when the computer can do everything for you. This would be a
dangerous attitude. We still have to understand what is going on, otherwise the
computer can in fact get us into trouble.

To illustrate this, use MATLAB to compute the inverse Laplace transform of
$\log s$. Now take the answer to this, and compute its Laplace transform. Tell what
happens, and where the mistake is.

6. The paradox in Exercise 5 has to do with integrating the Laplace transform.
To see this, suppose that $y(t)/t$ is of exponential type; in particular it does not
blow up at the origin. We want to find a formula for $\mathcal{L}(y(t)/t)$ in terms of
$\mathcal{L}(y(t))$. Let’s define $\tilde{y}(t) = y(t)/t$, and let $\tilde{F}(s)$ denote its Laplace transform.
Our objective is then to find a formula for $\tilde{F}(s)$. Using (2) you should be able
to get a fromula for $\tilde{F}'(s)$ in terms of $F(s)$. Now comes the subtle part. In
order to find $\tilde{F}$, we have to integrate $\tilde{F}.....$  but on what interval? Many of
the Laplace transforms we have seen blow up before we get to the origin, so we
cannot integrate from 0 to $s$. The only safe thing to do is to integrate from $s$
to $\infty$, and then we can use our observation from Exercise 2, that “$F(\infty) = 0$”.
Using this discussion, derive the formula

\begin{equation}
\mathcal{L}(y(t)/t) = \int_s^\infty F(s)ds.
\end{equation}

7. Use \texttt{laplace} to compute the Laplace transform of

\[ y(t) = \frac{e^{-1/(4t)}}{t^{3/2}}. \]

8. Show by calculation (not computer), that the same answer can be derived from
formula (3) and Exercise 3.