Test 1

Please note that there are some matrices and their echelon forms at the end of the test.

(1) Let $A$, $X$ and $B$ be as below.
(a) Find the general solution to the system $AX = B$. What are the translation and spanning vectors?
(b) Find a basis for the nullspace of $A$.
(c) Find a basis for the column space of $A$.

\[
A = \begin{bmatrix}
1 & 2 & 0 & 2 & 2 \\
2 & 4 & 3 & 1 & 1 \\
1 & 3 & 4 & 1 & 3
\end{bmatrix} \quad B = \begin{bmatrix}
2 \\
1 \\
3
\end{bmatrix} \quad X = \begin{bmatrix}
x \\
y \\
z \\
w \\
u
\end{bmatrix}.
\]

(2)
(a) Create a $4 \times 5$ matrix $A$ such that $AX = B$ is solvable if and only if $B$ belongs to the span of $[3, 0, 2, 0]^t$ and $[2, 0, 0, -3]^t$. Choose $A$ so that only 5 of its entries are 0.
(b) Explain why it is not possible to choose $A$ so that fewer than 5 entries are 0.

(3) Create a system of four equations in five unknowns (reader’s choice) such that the solution space is a plane in $\mathbb{R}^5$. Do not make any coefficients equal 0. Explain why your example works.

(4) Suppose that $X_1$, $X_2$ and $X_3$ are linearly independent elements of some vector space. Let $Y_1 = X_1 + 2X_2$, $Y_2 = X_1 + X_2$ and $Y_3 = X_1 + X_2 + X_3$. Prove that $Y_1$, $Y_2$ and $Y_3$ are also independent.

(5) Let $X$ and $Y$ be elements of some vector space. Let $U = X - Y$, $V = X + 2Y$ and $W = 3X - 7Y$. Which (if any) is bigger, span \{X, Y\} or span \{U, V, W\}? Prove your answer carefully. What are the possible dimensions for the latter space?

(6) Below are two sets of vectors in $\mathbb{R}^3$. One of these sets does not span $\mathbb{R}^3$. Which one does not span? Explain your answer in terms of dimension theory.

\[
S = \{[1, 2, -1]^t, [1, 1, 1]^t, [3, 4, 3]^t\} \\
T = \{[1, 2, -1]^t, [1, 1, 1]^t, [3, 4, 1]^t\}
\]

(7) Let $W$ be the set of matrices of the form below where $a$, $b$ and $c$ range over all real numbers.
(a) Prove that $W$ is a subspace of $M(1, 4)$. (Do not use the fact that spans are subspaces. Instead, reason directly from the definition of “subspace”.)
(b) Find a spanning set for $W$. What is the dimension of $W$? Prove your answer.
\[
[a + 2b + 5c, 2a + 6c, 3a - b + 8c, a + 4b + 8c]
\]
(8) Demonstrate your understanding of the test for independence by using it to test the following matrices for independence. (Other methods will not be accepted.) If they are dependent, use your work to express one of them in terms of the others. Find a basis for their span. What is the dimension of their span?

\[
\begin{bmatrix}
1 & 1 \\
4 & 3 \\
\end{bmatrix} \quad \begin{bmatrix}
2 & 1 \\
3 & 0 \\
\end{bmatrix} \quad \begin{bmatrix}
2 & 1 \\
2 & -1 \\
\end{bmatrix} \quad \begin{bmatrix}
1 & 1 \\
6 & 5 \\
\end{bmatrix}
\]

**Appendix:** Some matrices and their echelon forms.

\[
\begin{bmatrix}
1 & 1 & 4 & 3 \\
2 & 1 & 3 & 0 \\
2 & 1 & 2 & -1 \\
1 & 1 & 6 & 5 \\
\end{bmatrix} \quad \begin{bmatrix}
1 & 0 & 0 & -2 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 2 & 1 \\
1 & 1 & 1 & 1 \\
4 & 3 & 2 & 6 \\
3 & 0 & -1 & 5 \\
\end{bmatrix} \quad \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & -2 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 5 \\
2 & 0 & 6 \\
3 & -1 & 8 \\
1 & 4 & 8 \\
\end{bmatrix} \quad \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 3 & 1 \\
2 & 0 & -1 & 4 \\
5 & 6 & 8 & 8 \\
\end{bmatrix} \quad \begin{bmatrix}
1 & 0 & -1/2 & 0 \\
0 & 1 & 7/4 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]