Lesson 5  Section 1.9  Exact Equations

\[ M(x,y) \, dx + N(x,y) \, dy \]

Is an exact form if there exists a function \( F(x,y) \) such that

\[ M(x,y) \, dx + N(x,y) \, dy = dF = \frac{\partial F}{\partial x} (x,y) \, dx + \frac{\partial F}{\partial y} (x,y) \, dy \]

\[
\frac{\partial F}{\partial x} = M; \quad \frac{\partial F}{\partial y} = N
\]

Test for exactness. Let \( M(x,y) \) and \( N(x,y) \) and \( \frac{\partial M}{\partial y}, \frac{\partial N}{\partial x} \) be continuous in a region \( R \) of the plane which has no holes.

Then

\[ M \, dx + N \, dy \]

is exact if and only if

\[ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \]
Example: Determine whether this form is exact and find \( F \)

\[(1) \quad (y + 3x^2) \, dx + x' \, dy \]

\[\text{Sol: } M = y + 3x^2; \quad N = x\]

\[\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = 1.\]

Domain of \( M, N \) is \( \mathbb{R}^2 \) which is simply connected.

Find \( F \) such that

\[\frac{\partial F}{\partial x} = y + 3x^2; \quad \frac{\partial F}{\partial y} = x.\]

\[
\begin{align*}
\frac{\partial F}{\partial x} &= y + 3x^2; & F &= 2y + x^3 + \psi(y) \\
\frac{\partial F}{\partial y} &= x + \psi' = x; & \psi' &= 0, \\
\psi &= c.
\end{align*}
\]

\[F(x, y) = xy + x^3 + c\]
Solve the differential equation

\((y^2 + \cos x) \, dx + (2xy + \sin y) \, dy = 0\)

\(y(0) = \pi\)

\(M = y^2 + \cos x; \quad N = 2xy + \sin y\)

\(\frac{\partial M}{\partial y} = 2y; \quad \frac{\partial N}{\partial x} = 2y\)

\(\text{Domain: } \mathbb{R}^2 \text{ simply connected.}\)

\(\frac{\partial F}{\partial x} = y^2 + \cos x; \quad F(x,y) = xy^2 + \sin x + \int y \, dy\)

\(\frac{\partial F}{\partial y} = 2xy + \int y \, dy = 2xy + \sin y\)

\(4'(y) = \sin y; \quad y = -\cos y + C.\)

So

\[ F(x,y) = xy^2 + \sin x - \cos y + C. \]

\(C = \frac{\sin \pi}{\pi} + C.\)

Solution:

\[ xy^2 + \sin x - \cos y = C \]

\(x=0, \quad y = \pi; \quad -\cos \pi = C, \quad C = 1.\)

\[ xy^2 + \sin x - \cos y = 1 \]
Interpolating Factors

\[ M(x,y) \, dx + N(x,y) \, dy = 0 \]
\[ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \]

Is it possible to find a function \( I(x,y) \) such that

\[ I(x,y) \, M(x,y) \, dx + N(x,y) \, I(x,y) \, dy \]

is exact? In other words,

\[ \frac{\partial}{\partial y} \left( I \, M \right) = \frac{\partial}{\partial x} \left( I \, N \right) \]

\[ I \, \frac{\partial M}{\partial y} + M \, \frac{\partial I}{\partial y} = I \, \frac{\partial N}{\partial x} + N \, \frac{\partial I}{\partial x} \]

\[ I \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = N \, \frac{\partial I}{\partial x} - M \, \frac{\partial I}{\partial y} \]

This can be very complicated to solve.

**Smaller question**: Can we find \( I = I(x) \) ?

In this case,

\[ I(x) \left( \frac{\partial M}{\partial y} - \frac{2N}{\partial x} \right) = N \frac{2I}{\partial x} \]

\[ \frac{1}{I} \frac{dI}{dx} = \frac{\partial M}{\partial y} - \frac{2N}{\partial x} = \frac{Q(x)}{N} \text{ only} \]
\[ \frac{1}{I} \frac{dI}{dy} = \frac{\frac{\partial}{\partial x} (x+y^{-2}) - \frac{\partial}{\partial y} (x+y^{-2})}{M} = y(y^2) \]

**Example:**

\[ (3xy - 2y) \, dx + x(x+y^{-2}) \, dy = 0 \]

\[ y(1) = 3 \]

\[ M = 3xy - 2y \quad \text{and} \quad N = x^2 + xy^{-2} \]

\[ \frac{\partial M}{\partial y} = 3x + 2y^{-2} \quad \text{and} \quad \frac{\partial N}{\partial x} = 2x + y^{-2} \]

\[ \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = x + y^{-2} \]

\[ \frac{\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}}{N} = \frac{x + y^{-2}}{x^2 + xy^{-2}} = \frac{1}{x} \]

\[ \frac{1}{I} \frac{dI}{dx} = \frac{1}{x} \quad \text{so} \quad \ln |I| = \ln 12101 \]

\[ I = x . \]
Multiply equation by \( x \).

\[ x(3xy - 2y) \, dx + x^2(x + y^{-2}) \, dy = 0 \]

\[ M = 3x^2y - 2xy^{-1} \quad \text{and} \quad N = x^3 + x^2y^{-2} \]

\[ \frac{\partial M}{\partial y} = 3x^2 + 2x y^{-2} \quad \text{and} \quad \frac{\partial N}{\partial x} = 3x^2 + 2x y^{-2} \]

Find \( F \) such that

\[ \frac{\partial F}{\partial x} = 3x^2y - 2xy^{-1} \quad \text{and} \quad \frac{\partial F}{\partial y} = x^3 + x^2y^{-2} \]

\[ F = x^3y - x^2y^{-1} + \varphi(y) \]

\[ \frac{\partial F}{\partial y} = x^3 + x^2y^{-2} + \varphi'(y) = x^3 + x^2y^{-2} \]

\[ \varphi'(y) = 0 \]

Hence the solution is:

\[ F(x, y) = x^3y - x^2y^{-1} = c. \]

\[ y(1) = 3 \quad \text{and} \quad F(1, 3) = 3 - \frac{1}{3} = \frac{8}{3} \]

\[ x^3y - x^2y^{-1} = \frac{8}{3} \]
Solve:
\[ \frac{dy}{dx} + p(x)y = q(x) \]

by the method:
\[ \frac{dy}{dx} + p(x)y - q(x) = 0 \]

\[ dy + (p(x)y - q(x)) \, dx = 0 \]

\[ M(x, y) = p(x)y - q(x) ; \quad N = 1. \]

\[ \frac{\partial M}{\partial y} = p(x) ; \quad \frac{\partial N}{\partial x} = 0. \quad \text{Not exact} \]

Integrating factor:
\[ \frac{1}{I} \, \frac{dI}{dx} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = p(x) \]

\[ \frac{1}{I} \cdot \frac{dI}{dx} = p ; \quad \frac{d}{dx} \ln I = p(x) \]

\[ I = e^{\int p(x) \, dx} \]

\[ \int dy + I (p(x)y - q(x)) \, dx = 0 \quad \text{is exact} \]

\[ \frac{\partial}{\partial y} F(x, y) = I ; \quad F(x, y) = y \, I + \psi(x) \]
\[ \frac{dF}{dx} = \frac{d}{dx}(y I) + \frac{d}{dx}(2\pi) = (p(x) y - q(x)) I. \]

\[ \frac{dI}{dx} = p I \]

\[ \frac{d\pi}{dx} = -q(x) I. \]

\[ I \ y - \int q(x) I(x) \, dx = c. \]