Write your name in the top left corner. Attempt all questions. You must show all work in order to obtain credit.

1. (a) (5 pts) For each of the following statements state if it is true or false (no justification required).
   i. If \( B = \{v_1, v_2, \ldots, v_n\} \) is a basis of a vector space \( V \), then the set \( C = \{-v_1, -v_2, \ldots, -v_n\} \) is also a basis of \( V \).
   ii. If \( B = \{v_1, v_2, \ldots, v_n\} \) is a basis of a vector space \( V \), and \( C = \{w_1, \ldots, w_n\} \) is a basis of \( V \), then \( D = \{v_1 + w_1, \ldots, v_n + w_n\} \) is again a basis of \( V \).
   iii. If \( B = \{v_1, \ldots, v_n\} \) is a basis of \( V \), then any subset of \( B \) is again a basis of \( V \).
   iv. If \( B = \{v_1, \ldots, v_n\} \) is a basis of \( V \), then any set of vectors properly containing \( B \) is linearly dependent.
   v. If \( B = \{v_1, \ldots, v_n\} \) is a basis of \( V \), then any set of vectors properly containing \( B \) must span \( V \).

(b) (5 pts) Let \( P_3 \) be the vector space of all real polynomials in \( X \) of degree at most 3 and let \( V \subset P_4 \) be the subspace consisting of polynomials which vanish at \( X = 1 \) and whose derivative vanish at \( X = -1 \). Compute a basis for the subspace \( V \).
2. (a) (5pts) Let $S$ and $T$ be two finite subsets of vectors of a vector space $V$ and suppose that $S$ is linearly dependent and $T$ linearly independent. For each one of the following subsets of $V$ state whether it they are linearly dependent or independent or that we do not have enough information to decide.

i. $S \cap T$: 

ii. $S \cup T$: 

iii. $S \setminus T$ (i.e. the set of elements of $S$ that are not in $T$): 

iv. $T \setminus S$ (i.e. the set of elements of $T$ that are not in $S$): 

v. $T \cup \{0\}$: 

(b) (5pts) Let $V = P_3$ equipped with the inner product defined by $(f, g) = \int_0^1 f(X)g(X)\,dX$. space and $\{f, g, h\}$ be an orthonormal set of vectors. Compute $||f + g + h||^2$. 

3. Let $S = \{1, (X + 1), (X + 1)^2, (X + 1)^3\}$ and $T = \{1, (X - 1), (X - 1)^2, (X - 1)^3\}$ be two bases of $P_3$.

(a) (3 pts) Compute the co-ordinates of the polynomial $v = X$ with respect to the basis $S$ i.e. compute the column vector $[v]_T$.

(b) (4 pts) Compute the transition matrix $P_{S\leftarrow T}$ for going from the basis $T$ to basis $S$.

(c) (3 pts) Using your answers to the previous part compute the column vector $[v]_S$ (i.e. the co-ordinate vector of $v = 1 + X + X^2 + X^3$ with respect to the basis $S$.)
4. (a) (5pts) Let $V = \mathbb{R}^3$ with the usual Euclidean inner product. Find an orthonormal basis for the subspace $W$ of $V$ consisting of all vectors of the form
\[
\begin{bmatrix}
a \\
a + b \\
b
\end{bmatrix}.
\]

(b) (5 pts) Let $W$ be the subspace of $\mathbb{R}^3$ (with Euclidean inner product) with orthonormal basis
\[
w_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad w_2 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ \frac{2}{\sqrt{5}} \end{bmatrix}.
\]

Write the vector
\[
v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\]
as $w + u$ with $w \in W$ and $u \in W^\perp$. 

5. (10pts) Find an orthonormal basis for the solution space of the homogeneous system

\[
\begin{bmatrix}
1 & 1 & -1 \\
2 & 1 & 3 \\
1 & 2 & -6
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= \begin{bmatrix} 0 \\
0 \\
0 \end{bmatrix}.
\]