Write your name in the top left corner. Attempt all questions. You must show all work in order to obtain credit.

1. (a) (3pts) Let $A$ be an $n \times n$ matrix. Define $\det(A)$.

(b) (3pts) Let $A = [a_{ij}]_{1 \leq i, j \leq 3}$ be a $3 \times 3$ matrix. Write down the full expression of $\det(A)$ in terms of the $a_{ij}$’s using the definition of $A$. (Be careful about the signs in front of different terms).

(c) (4pts) Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \\ -1 & 0 & 1 & 8 \end{bmatrix}.$$ 

Compute using row reductions the determinant of $A$. 


2. Let

\[ A = \begin{bmatrix}
1 & 1 & 1 \\
-1 & 0 & 1 \\
1 & 0 & 1
\end{bmatrix}. \]

(a) (3 pts) Compute all the cofactors of the matrix \( A \).

(b) (3 pts) Compute \( \det(A) \) using the expansion by co-factor formula.

(c) (4 pts) Compute \( A^{-1} \) using the adjoint formula.
3. (a) Let $E$ be the $3 \times 3$ elementary matrix corresponding to the row operation of exchanging row 1 and row 3.
   
i. (1 pt) Write down $E$ explicitly.

   ii. (1 pt) What is $\det(E)$?

   iii. (2 pts) Is the inverse of $E$ an elementary matrix? Justify your answer.

(b) (6 pts) Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$ 

Express $A$ as $A = E_1 \cdots E_m C$, where $C$ is in the reduced row echelon form and each $E_i$ is an elementary matrix.
4. (10 pts) Consider the system of linear equations given by $Ax = b$ where

\[
A = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
-1 & 0 & 1 & 2 & 3 \\
1 & 0 & 1 & 4 & 9 
\end{bmatrix},
\]

and

\[
b = \begin{bmatrix}
1 \\
2 \\
3 
\end{bmatrix}.
\]

Describe the solutions of the system.
5. Let $V$ be the set of $3 \times 3$ matrices.

(a) (2 pts) Is $V$ a vector space (where the operation $\oplus$ is the usual matrix addition and the operation $\odot$ is the usual scalar multiplication for matrices)? Justify.

(b) (2 pts) Let $U \subset V$ be defined by

$$U = \{ M \in V | \det(M) = 0 \}$$

(that is $U$ is the set of $3 \times 3$ matrices whose determinant is 0). Is $U$ a vector space (under the same operations as before)? Justify.

(c) (2 pts) Let $W \subset V$ be the set of all upper-triangular matrices. Is $W$ a vector space? Justify.

(d) (2 pts) Let $T \subset V$ be the set of matrices in row echelon form. Is $T$ a vector space? Justify.

(e) (2 pts) Let $P \subset V$ be the set of invertible matrices. Is $P$ a vector space? Justify.
6. (extra credits)

(a) (5 pts) Let $A$ be an $n \times n$ matrix such that $A^3 = 0$. Prove that $\det(I_n - A) \neq 0$ (where $I_n$ is the $n \times n$ identity matrix).

(b) (5 pts) Let $Q$ be the $n \times n$ matrix in which each entry is 1. Show that $\det(Q - nI_n) = 0$. 