QUADRATIC UNIPOTENT ARTHUR PARAMETERS AND RESIDUAL SPECTRUM OF SYMPLECTIC GROUPS

By HENRY H. KIM and FREYDOON SHAHIDI

Abstract. The purpose of this paper is to study certain quadratic unipotent Arthur parameters in the sense of Moeglin and use them to parametrize a part of the residual spectrum of symplectic groups over number fields, coming from the conjugacy class of Borel subgroups. In particular, using certain identities satisfied by local intertwining operators, Arthur’s multiplicity formula is established for them which remarkably enough appears by itself in the corresponding residue of the Eisenstein series.

Introduction. In this paper we study certain quadratic unipotent Arthur parameters in the sense of Moeglin [23] (cf. Section 3) and realize them as Arthur parameters [1, 2] for certain square integrable residues of Eisenstein series attached to the conjugacy class of Borel subgroups of a symplectic group over a number field. Consequently, using certain local identities proved in [11], we prove Arthur’s multiplicity formula for them (Theorem 5.1), which remarkably enough appears by itself in the corresponding residue of the Eisenstein series.

More precisely, let \( G = \text{Sp}_{2n} \) over a number field \( F \) with ring of adeles \( \mathbb{A}_F \). As in [23], we use \( \hat{G} = O_{2n+1}(\mathbb{C}) \) to denote its dual group. Let \( \mu_1, \ldots, \mu_k \) be \( k \) distinct nontrivial quadratic grossencharacters of \( F \). Fix integers \( r_1 \geq \ldots \geq r_k \geq 2 \) with \( r_1 + \ldots + r_k \leq n \) and choose \( r_0 \) such that \( r_0 + r_1 + \ldots + r_k = n \). Then \( \chi = \chi(\mu_1, \ldots, \mu_1, \ldots, \mu_k, \ldots, \mu_k, 1, \ldots, 1) \) defines a character of \( T(F) \backslash T(\mathbb{A}_F) \), where \( T \) is the subgroup of diagonal elements in \( G \). An Eisenstein series [18] attached to a character of \( T(\mathbb{A}_F) \) will contribute to the residual spectrum only if the character is of the above type (Proposition 4.6).

By [14, 16], the character \( \chi \) defines a homomorphism of the Weil group \( W_F \) into a Cartan subgroup of \( \text{SO}_{2n+1}(\mathbb{C}) \). Composing this homomorphism with the standard action of \( O_{2n+1}(\mathbb{C}) \) on \( \mathbb{C}^{2n+1} \) will then give a completely reducible representation of \( W_F \) on \( \mathbb{C}^{2n+1} \) which decomposes according to eigenvalues \( \mu_1, \ldots, \mu_k, \) and \( 1 \), with multiplicities \( 2r_1, \ldots, 2r_k, \) and \( 2r_0 + 1 \), respectively. Write \( \mathbb{C}^{2n+1} = V_0 \oplus V_1 \oplus \ldots \oplus V_k \), where each \( V_i, \dim V_i = 2r_i \), is the eigenspace attached to
eigenvalue $\mu_i$, $1 \leq i \leq k$, and $V_0$ is the trivial eigenspace of dimension $2r_0 + 1$. In this way we get an embedding of $\prod_{i=0}^k O(V_i) \subset O_{2n+1}(\mathbb{C})$.

Now, for each $i$, $1 \leq i \leq k$, let $O_i$ be the unipotent orbit of $O(V_i)$ attached to the principal Jordan block $(2r_i - 1, 1)$. Let $O_o$ be the principal unipotent orbit of $O(V_0)$, i.e. the one attached to the Jordan block $(2r_0 + 1)$.

The Arthur parameter of interest to us is a homomorphism

$$\psi: W_F \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \to O_{2n+1}(\mathbb{C})$$

which factors through $\prod_{i=0}^k O(V_i)$, sends $W_F$ to the center of $\prod_{i=0}^k O(V_i)$ according to $1 \otimes \mu_1 \otimes \ldots \otimes \mu_k$, is trivial on the middle $SL_2(\mathbb{C})$, and for which

$$\psi(1, 1, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix})$$

belongs to $\prod_{i=0}^k O_i$. This is clearly a quadratic unipotent Arthur parameter in the sense of Moeglin [23] (see Section 3). To $\psi$, Arthur associates a Langlands’ parameter $\phi_{\psi}$ (see Section 3).

In this paper, we use Langlands theory of Eisenstein series [18] to construct the representations in $\Pi_{\phi_{\psi}}$ as residues of the Eisenstein series associated to the character $\chi$ (Theorems 4.5 and 5.1). Using certain identities satisfied by local intertwining operators which was proved in [11], we then verify Arthur’s multiplicity formula for these square integrable residues (Theorem 5.1). (See section 3 for Arthur’s multiplicity formula.) It is remarkable that in fact the formula itself appears in the corresponding residue of the Eisenstein series. We note that the local $R$-group $C_{\phi_{\psi}}$ for the parameter $\phi_{\psi}$ (see Section 3) is the Knapp-Stein $R$-group of the unitary principal series $I_\psi = Ind_{B_0(F)}^{M(F)} \chi_\psi$, where $M$ is the Levi-subgroup whose $L$-group is $M^* = Cent (im \phi_{\psi}^+, G^+)$.

The technical combinatorial part of dealing with the residues of Eisenstein series (Proposition 4.4 and Theorem 4.5) which is an important step in the proof is contained in Section 4. They rely on several technical lemmas about Weyl groups and normalizing factors (Lemmas 4.7 and 4.8). The final interpretation of Theorem 4.5 and the proof of Arthur’s multiplicity formula (Arthur’s condition in the language of Moeglin [23]) is done in Section 5 (Theorem 5.1). In particular, in Section 5 we determine Arthur parameters of the square integrable residual spectrum of $Sp_4(\mathbb{A}_F)$ coming from the conjugacy class of Borel subgroups. Using [12], this implies the exhaustion, i.e., that quadratic unipotent Arthur parameters completely determine the residual spectrum of $Sp_4(\mathbb{A})$ coming from this conjugacy class and conversely.

We expect that quadratic unipotent Arthur parameters completely parametrize all the residual spectrum of $Sp_{2n}$ coming from the conjugacy class of Borel subgroups. When these residues are unramified, i.e., when $\chi = 1$, the problem has been completely solved by Moeglin [22, 20]. We expect that her results will
play an important role in proving the exhaustion in general. We would like to thank her for patiently answering many of our questions [21].

Finally in view of [23], we expect that the method of the present paper can be equally well applied to the case of orthogonal groups.

1. Preliminaries. Let $F$ be a field and let $G = \text{SO}_{2n+1}, \text{Sp}_{2n}$ or $\text{SO}_{2n}$ over $F$. Let $J_n$ be the $n \times n$ matrix given by

$$J_n = \begin{pmatrix} \ & \ & \ & 1 \\ \ & \ & 1 \\ \ & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}.$$ 

Let $J_{2n}' = \begin{pmatrix} I_n & J_n \\ -J_n & I_n \end{pmatrix}$. Then

$$\text{Sp}(2n) = \{ g \in GL(2n) | \ gJ_{2n}'g = J_{2n}' \} ,$$

and

$$\text{SO}(n) = \{ g \in GL(n) | \ gJ_ng = J_n; \det(g) = 1 \} .$$

In each case we let $T$ be the maximal split torus consisting of diagonal matrices in $G$. Then

$$T(F) = \left\{ \begin{pmatrix} l_1 \\ l_2 \\ \cdot \cdot \cdot \\ l_n \\ l_n^{-1} \\ \cdot \cdot \cdot \\ l_2^{-1} \\ l_1^{-1} \end{pmatrix} \mid l_i \in F^* \right\} ,$$
if $G = \text{Sp}(2n)$ or $\text{SO}(2n)$, and

$$T(F) = \left\{ \begin{pmatrix} l_1 & & & & & & \\ & l_2 & & & & & \\ & & \ddots & & & & \\ & & & \ldots & & & \\ & & & & 1 & & \\ & & & & & \ldots & \\ & & & & & & l_1^{-1} \\ l_n & & & & \ldots & & l_n^{-1} \end{pmatrix} \middle| l_i \in F^* \right\}$$

if $G = \text{SO}(2n + 1)$.

Let $\Phi(G, T)$ be the roots of $G$ with respect to $T$. We choose the ordering on the roots so that the Borel subgroup $B$ is the subgroup of upper triangular matrices in $G$. Let $\Delta$ be the simple roots in $\Phi(G, T)$ given by $\Delta = \{ \alpha_j \}_{j=1}^n$, with $\alpha_j = e_j - e_{j+1}$ for $1 \leq j \leq n - 1$, and

$$\alpha_n = \begin{cases} 
2e_n & G = \text{Sp}(2n), \\
e_{n-1} + e_n & G = \text{SO}(2n).
\end{cases}$$

We let $\langle , \rangle$ be the standard Euclidean inner product on $\Phi(G, T)$. If $\Phi$ is a root system of type $B_n$, $C_n$, or $D_n$, then we denote by $G(\Phi)$ the split group with root system $\Phi$.

For $G = \text{SO}(2n + 1)$ or $\text{Sp}(2n)$, the Weyl group $W(G/T) \cong S_n \ltimes \mathbb{Z}_2^n$. $S_n$ acts by permutations on the $\lambda_i$, $i = 1, \ldots, n$. We will use standard cycle notation for the elements of $S_n$. Thus $(ij)$ interchanges $\lambda_i$ and $\lambda_j$. If $c_i$ is the nontrivial element in the $i$th copy of $\mathbb{Z}_2$ then $c_i$ takes $\lambda_i$ to $\lambda_i^{-1}$. The element $c_i$ is called a sign change because its action on $\Phi(G, T)$ takes $e_i$ to $-e_i$. For $G = \text{SO}(2n)$, the Weyl group is given by $W(G/T) \cong S_n \ltimes \mathbb{Z}_2^{n-1}$. $S_n$ acts by permutations on the $\lambda_i$, and $\mathbb{Z}_2^{n-1}$ acts by even numbers of sign changes. The requirement that the number of sign changes be even comes from the determinant condition in $\text{SO}(2n)$. Note that the sign change $c_i$ is an element of $O(2n)$ and normalizes $T(F)$. Each $c_i$ acts on $\text{SO}(2n)$ by conjugation, and $c_n$ induces the nontrivial graph automorphism on the Dynkin diagram of $\Phi(G, T)$.

2. Unipotent orbits of classical groups over $\mathbb{C}$. The theory of Jordan normal forms implies that a unipotent matrix in $GL_N$ is conjugate to $J(p_1) \oplus J(p_2) \oplus \cdots \oplus J(p_s)$, where $p_1 \geq p_2 \geq \cdots \geq p_s$, $p_1 + p_2 + \cdots + p_s = N$, and $J(p)$ is the $p \times p$ Jordan matrix with entries 1 just above the diagonal and the diagonal, and zero everywhere else. Therefore, unipotent classes in $GL_N$ are in 1 to 1
correspondence with partitions \( \lambda \) of \( N \). We use the following standard notation for \( \lambda \): \( \lambda = (1^{r_1}, 2^{r_2}, 3^{r_3}, \ldots) \), where \( r_j \) is the number of \( p_i \) equal to \( j \).

Let \( G \) be a classical group, of type \( B_n \) (\( O_{2n+1}(\mathbb{C}) \)), \( C_n \) (\( \text{Sp}_{2n}(\mathbb{C}) \)) or \( D_n \) (\( O_{2n}(\mathbb{C}) \)). We start with the following facts:

1. \( X, X' \in G \) are conjugate in \( G \) if and only if they are conjugate in \( GL_N, N = 2n + 1 \) or \( 2n \).
2. Let \( X \in GL_N \) be unipotent. Then \( X \) is conjugate to an element of \( G \) if and only if \( r_i \) is even for even \( i \) in the orthogonal case and for odd \( i \) in the symplectic case.

Therefore for \( G = O_{2n+1}(\mathbb{C}) \), unipotent classes are in 1 to 1 correspondence with partitions \( \lambda \) of \( 2n + 1 \) such that \( r_i \) is even for even \( i \).

Let \( u \) be a unipotent element in \( G \) and let \( S_u \) be its centralizer in \( G \). Then we have:

3. In the orthogonal case (resp. symplectic) case, \( S_u / S^0_u \) is \( k \) product of \( \mathbb{Z}/2\mathbb{Z} \), where \( k \) is the number of odd (resp. even) \( i \) such that \( r_i > 0 \).

Here we note that for \( G = GL_N(\mathbb{C}) \), the centralizer \( Z_G(S) \) is connected for any subset \( S \) of \( G \).

We say that a unipotent element \( u \) is distinguished if all maximal tori of \( \text{Cent}(u, G) \) are contained in the center of \( G^0 \), the connected component of the identity. This is equivalent to the fact that the unipotent orbit \( O \) of \( u \) does not meet any proper Levi subgroup of \( G \) (Spaltenstein [30, p. 67]; i.e., if \( L \) is a Levi subgroup of a parabolic subgroup of \( G \) and \( u \in L \) for a \( u \in O \), then \( L^0 = G^0 \)). If \( G = O_{2n+1}(\mathbb{C}) \), then \( G^0 = SO_{2n+1}(\mathbb{C}) \) and \( G^0 \) has trivial center. By Carter [5], for \( G = O_{2n+1}(\mathbb{C}) \) or \( O_{2n}(\mathbb{C}) \), if \( u \) is a unipotent element with Jordan blocks \( (1^{r_1}, 2^{r_2}, \ldots) \), then the reductive part of the connected centralizer \( \text{Cent}(u, G)^0 \) is of type

\[
\prod_{i \text{ even}} C_{r_i/2} \times \prod_{i \text{ odd}, r_i \text{ even}} D_{r_i/2} \times \prod_{i \text{ odd}, r_i \text{ odd}} B_{(r_i-1)/2}.
\]

Therefore, \( O \) is a distinguished unipotent class if and only if it has Jordan blocks \( (1^{r_1}, 3^{r_3}, 5^{r_5}, \ldots) \), where \( r_i = 0 \) or \( 1 \).

**JACOBSON-MOROZOV THEOREM.** Suppose \( u \) is a unipotent element in a semi-simple algebraic group \( G \). Then there exists a homomorphism \( \phi : SL_2 \rightarrow G \) such that \( \phi \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) = u \).

Here, replacing \( \phi \) by a conjugate under \( G \), we can assume that \( \phi \left( \begin{array}{cc} a & 0 \\ 0 & a^{-1} \end{array} \right) \) is in the closure of the positive Weyl chamber in the maximal torus. In fact, by the theory of weighted Dynkin diagrams (cf. Section 5.6 of [5]), \( \phi \left( \begin{array}{cc} a & 0 \\ 0 & a^{-1} \end{array} \right) \) is uniquely determined by the unipotent orbit of \( u \) as follows (Carter [5, p. 395]):
Suppose $O$ has Jordan blocks $(d_1, d_2, d_3, \ldots)$. For each $d_i$, we take the set of integers $d_i - 1, d_i - 3, \ldots, 3 - d_i, 1 - d_i$. We then take the union of these sets for all $d_i$ and write this union as $(\xi_1, \xi_2, \xi_3, \ldots)$ with $\xi_1 \geq \xi_2 \geq \xi_3 \geq \ldots$. Then

$$\phi \left( \begin{array}{cc} a & 0 \\ 0 & a^{-1} \end{array} \right) = \text{diag} (a^{\xi_1}, a^{\xi_2}, a^{\xi_3}, \ldots).$$

**Lemma.** ([3, Proposition 2.4]) Let $u$ be a unipotent element and $\phi: SL_2 \longrightarrow G$ be a homomorphism such that $\phi \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) = u$. Let $S_\phi = \text{Cent} (\text{im} \phi, G) \subset S_u = \text{Cent} (u, G)$ and $U^u$ be the unipotent radical of $S_u.$ Then

1. $S_u = S_\phi \cdot U^u$, a semi-direct product. $S_\phi$ is reductive.
2. The inclusion $S_\phi \subset S_u$ induces an isomorphism between $S_\phi/Z_G$ and $S_u/Z_G$.

**3. Quadratic unipotent Arthur parameters.** We follow Moeglin [23]. Let $F$ be a number field and let $W_F$ be the global Weil group of $F$. For $G = \text{Sp}_{2n}$, we can take the dual group $G^* = O_{2n+1}(\mathbb{C})$. An Arthur parameter is a homomorphism $\psi: W_F \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \longrightarrow O_{2n+1}(\mathbb{C})$ with the following properties: (The usual definition of Arthur parameter uses Langlands’ hypothetical group $L_F$. But since we are only dealing with Langlands’ quotients which come from principal series, $W_F$ is enough.)

1. $\psi(W_F)$ is bounded and included in the set of semi-simple elements of $G^*$.
2. The restriction of $\psi$ to the 2 copies of $SL_2(\mathbb{C})$ is algebraic.
3. Composing $\psi|_{W_F}$ with the determinant of $G^*$ gives a quadratic character of $W_F$, denoted by $\text{det} \psi$. We want $\text{det} \psi = 1$.

We call an Arthur parameter quadratic unipotent if the following two conditions are satisfied:

1. $(\psi|_{SL_2(\mathbb{C}) \times 1}) \equiv 1$;
2. $\psi|_{W_F}$ is trivial on the intersection of the kernels of the quadratic characters of $W_F$.

Because of conditions (1) and (5), the action of $\psi(W_F)$ gives an orthogonal decomposition:

$$\mathbb{C}^{2n+1} = V_0 \oplus V_1 \oplus \cdots \oplus V_k,$$

where $\dim V_0 = 2r_0 + 1$, $\dim V_i = 2r_i$, $2r_0 + 1 + 2r_1 + \cdots + 2r_k = 2n + 1$, $r_1 \geq \cdots \geq r_k$ and $V_i$ is the eigenspace with eigenvalue $\mu_i$. Here $\mu_1, \ldots, \mu_k$ are nontrivial distinct quadratic grøssencharacters of $F$, viewed as characters of $W_F$ (cf. [14, 16]), and $\dim V_i$ being even comes from condition (3).
The parameter $\psi$ factors through $\prod_{i=0}^{k} O(V_i)$:

$$\psi: W_F \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \longrightarrow \prod_{i=0}^{k} O(V_i).$$

(1) $W_F$ is mapped into the product of centers of $O(V_i)$

$$\psi|_{W_F}: w \mapsto 1 \times \mu_1(w) \times \cdots \times \mu_k(w) \in \{\pm 1\} \times \{\pm 1\} \times \cdots \times \{\pm 1\},$$

where $\{\pm 1\}$ is the center of $O(V_i)$, for $i = 0, \ldots, k$.

(2) By Jacobson-Morozov theorem, $\psi|_{1 \times 1 \times SL_2(\mathbb{C})}$ defines a unipotent orbit of $G^*$ of the form

$$\prod_{i=0}^{k} O_i,$$

where $O_i$ is a unipotent orbit of $O(V_i)$. Inside $O_i$ we fix an element $u_i$ such that

$$\psi \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \prod_{i=0}^{k} u_i.$$

Let $S_\psi = \text{Cent}(im\psi, G^*)$ and

$$C_\psi = S_\psi / S_\psi^0 Z_{G^*}.$$

We know that $S_\psi$ is a maximal reductive subgroup of $\prod_{i=0}^{k} \text{Cent}(u_i, O(V_i))$. Therefore $S_\psi^0 = 1$, i.e., $S_\psi$ is finite if and only if each $u_i$ is a distinguished unipotent element in $O(V_i)$. Especially, since $O_2(\mathbb{C})$ has no distinguished unipotent element, we have

**Lemma.** Let $\psi$ be a quadratic unipotent Arthur parameter. Suppose $S_\psi^0 = 1$. Then $r_k \geq 2$.

Now it is clear that $S_\psi / S_\psi^0 Z_{G^*}$ is equal to

$$\text{Cent}(u_0, O(V_0)) / \text{Cent}(u_0, O(V_0))^0 Z_{O(V_0)} \prod_{i=1}^{k} \text{Cent}(u_i, O(V_i)) / \text{Cent}(u_i, O(V_i))^0.$$

Here $\text{Cent}(u_i, O(V_i)) / \text{Cent}(u_i, O(V_i))^0$ is $t$ product of $\mathbb{Z} / 2\mathbb{Z}$, where $t$ is the number of $i$ odd with $r_i > 0$ in Jordan blocks.

For each place $v$ of $F$, we have a map $\psi_v = \psi|_{W_{F_v} \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C})}$. As in the global case, we can then define $S_{\psi_v}$. But in the local case, $\mu_{iv}$ may not be
distinct. Suppose $\mu_{1v} = \mu_{2v}$. Then in the above formula,

$$\text{Cent}(u_1, O(V_1))/ \text{Cent}(u_1, O(V_1))^0 \times \text{Cent}(u_2, O(V_2))/ \text{Cent}(u_2, O(V_2))^0$$

must be replaced by

$$\text{Cent}(u_1 \times u_2, O(V \oplus V_2))/ \text{Cent}(u_1 \times u_2, O(V_1 \oplus V_2))^0.$$ 

To any Arthur parameter $\psi$, Arthur associates a Langlands’ parameter $\phi_\psi$: $W_F \times SL_2(\mathbb{C}) \hookrightarrow G^*$ as follows:

$$\phi_\psi(w, 1) = \psi \left( w, 1, \begin{pmatrix} |w|^\frac{1}{2} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix} \right), \quad \phi_\psi|1 \times SL_2(\mathbb{C}) \equiv 1.$$ 

For quadratic unipotent Arthur parameter $\psi$, $\phi_\psi$ is given by

1. $\phi_\psi|SL_2(\mathbb{C}) \equiv 1$;
2. $\phi_\psi(w) = \prod_{i=0}^k \phi_{\psi_i}(w) \in \prod_{i=0}^k O(V_i)$, where each $\phi_{\psi_i}(w)$ is the associated Langlands parameter for $\psi_i$: $W_F \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \hookrightarrow O(V_i)$.

Now we recall Arthur’s conjecture. Let $S_{\phi_\psi} = \text{Cent}(im \phi_\psi, G^*)$ and

$$C_{\phi_\psi} = S_{\phi_\psi}/S_{\phi_\psi}^0 Z_{G^*}.$$ 

For each place $v$ of $F$, we have local Arthur parameters $\psi_v = \psi|W_{F_v} \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$, as well as $S_{\psi_v}$, $C_{\psi_v}$, $S_{\phi_{\psi_v}}$ and $C_{\phi_{\psi_v}}$. For each $v$, there is also a natural map $C_{\psi_v} \hookrightarrow C_{\psi_v}$ and a natural surjective $C_{\psi_v} \twoheadrightarrow C_{\phi_{\psi_v}}$. The parameter $\phi_{\psi_v}$ gives a $L$-packet $\Pi_{\phi_{\psi_v}}$ which consists of Langlands’ quotients.

It is a part of Arthur’s local conjecture [1, 2] that for each place $v$ of $F$, there is a pairing $\langle \cdot, \cdot \rangle$ on $C_{\phi_{\psi_v}} \times \phi_{\psi_v}$ and an enlargement $\Pi_{\phi_{\psi_v}}$ of $\Pi_{\phi_{\psi_v}}$ which allows an extension of $\langle \cdot, \cdot \rangle$ to $C_{\psi_v} \times \Pi_{\psi_v}$ such that $\pi \in \Pi_{\phi_{\psi_v}} \subset \Pi_{\psi_v}$ if and only if the function $\langle \cdot, \pi \rangle$ lies in the image of $C_{\phi_{\psi_v}}$ in $C_{\psi_v}$.

We define the global Arthur packet $\Pi_{\psi}$ to be the set of irreducible representations $\pi = \otimes_v \pi_v$ of $G(\mathbb{A})$ such that for each $v$, $\pi_v$ belongs to $\Pi_{\psi_v}$. Define the global pairing on $C_{\psi} \times \Pi_{\psi}$ by

$$\langle x, \pi \rangle = \prod_v \langle x_v, \pi_v \rangle,$$

for $\pi = \otimes_v \pi_v \in \Pi_{\psi}$ and $x \in C_{\psi}$ with image $x_v$ in $C_{\psi_v}$. 

Arthur’s conjecture (Global).

1. The representations in the packet corresponding to $\psi$ may occur in the discrete spectrum if and only if $S_{\psi}$ is finite, i.e., $S_{\psi}^0 = 1$. We call such an Arthur parameter elliptic.
(2) For an elliptic Arthur parameter $\psi$, there is a positive integer $d_\psi$ and a homomorphism $\epsilon_\psi: C_\psi \rightarrow \{\pm 1\}$ such that the multiplicity with which any $\pi \in \Pi_\psi$ occurs discretely in $L^2(G(F) \backslash G(\mathbb{A}))$ is

\[
\frac{d_\psi}{|C_\psi|} \sum_{x \in C_\psi} \epsilon_\psi(x)(x, \pi).
\]

For quadratic unipotent Arthur parameters, we have

**Lemma.** (Moeglin [23]) For $\psi$ quadratic unipotent, $\epsilon_\psi$ is trivial.

In Section 4, we restrict ourselves to the case where the unipotent orbits $O_i$ have Jordan blocks $(2r_i - 1, 1)$ for $i = 1, \ldots, k$ and $(2r_0 + 1)$ for $i = 0$, i.e., the ones with the most weighted Dynkin diagrams (cf. [5]). We will construct representations in $\Pi_\phi$ as residues of Eisenstein series associated to the character

$\chi = \chi(\mu_1, \ldots, \mu_1, \ldots, \mu_k, \ldots, \mu_k, 1, \ldots, 1)$, where $r_1 \geq r_2 \geq \cdots \geq r_k$ and $\mu_i$'s are mutually distinct and quadratic grössencharacters.

In Section 5, we interpret the result of Section 4 in terms of Arthur parameters and prove the multiplicity formula (3.1).

4. Residual spectrum of $\text{Sp}_{2n}$. We fix a nontrivial additive character $\eta = \otimes \eta_v$ of $\mathbb{A}/F$ and let $\xi(z, \mu)$ be the Hecke $L$-function with the ordinary $\Gamma$-factor so that it satisfies the functional equation $\xi(z, \mu) = \epsilon(z, \mu)\xi(1-z, \mu^{-1})$, where $\epsilon(z, \mu) = \prod_v \epsilon(z, \mu_v, \eta_v)$ is the usual $\epsilon$-factor (see [8, p. 159]). If $\mu$ is the trivial character $\mu_0$, then we write simply $\xi(z)$ for $\xi(z, \mu_0)$. We have the Laurent expansion of $\xi(z)$ at $z = 1$:

\[
\xi(z) = \frac{c(F)}{z-1} + a + \cdots.
\]

Let $\alpha^\vee$ be the coroot corresponding to $\alpha \in \Phi^+(G, T)$. Explicitly, for $\alpha = e_i - e_j$, $\alpha^\vee(\lambda) = t(1, \ldots, \overset{i}{\lambda}, \ldots, \overset{j}{\lambda}, \ldots, 1)$ is $T(F)$ for $1 \leq i < j \leq n$. For $\alpha = e_i + e_j$, $\alpha^\vee(\lambda) = t(1, \ldots, \overset{i}{\lambda}, \ldots, \overset{j}{\lambda}, \ldots, 1)$, for $1 \leq i < j \leq n$. For $\alpha = 2e_i$, $\alpha^\vee(\lambda) = t(1, \ldots, \overset{i}{\lambda}, \ldots, 1)$ for $1 \leq i \leq n$. Here dots represent 1.

Let $X(T)_F$ (resp. $X^*(T)_F$) be the group of $F$-rational characters (resp. cocharacters) of $T$. There is a natural pairing $\langle ., . \rangle: X(T)_F \times X^*(T)_F \rightarrow \mathbb{Z}$. For $\alpha, \beta \in \Phi(G, T)$, $\langle \beta, \alpha^\vee \rangle = 2(\beta, \alpha)/(\alpha, \alpha)$, where $(\ , \ )$ is the standard inner product in $\Phi(G, T)$. Let $\omega_1 = e_1 + \cdots + e_1$. Then $\omega_1, \ldots, \omega_n$ are the fundamental weights of $G$ with respect to $(G, T)$. Since $G$ is simply connected, $X(T)_F = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n$, and $X^*(T)_F = \mathbb{Z}\alpha_1^\vee + \cdots + \mathbb{Z}\alpha_n^\vee$. Set $\alpha^* = X(T)_F \otimes \mathbb{R}$, $\alpha^*_\mathbb{C} = X(T)_F \otimes \mathbb{C}$, and $\alpha = X^*(T)_F \otimes \mathbb{R} = \text{Hom}(X(T)_F, \mathbb{R})$, $\alpha_\mathbb{C} = X^*(T)_F \otimes \mathbb{C}$. The positive Weyl chamber
in $\mathfrak{a}^*$ is

$$
C^+ = \{ \Lambda \in \mathfrak{a}^* | \langle \Lambda, \alpha^\vee \rangle > 0, \text{ for all } \alpha \text{ positive roots} \}
= \left\{ \sum_{i=1}^{n} a_i \omega_i | a_i > 0 \right\}.
$$

Let $B = TU$ be the Borel subgroup, where $U$ is the unipotent radical. Let $K_\infty$ be the standard maximal compact subgroup of $G(A_\infty)$ and for $v$ finite, let $K_v = G(O_v)$, where $O_v$ is the ring of integers of $F$. Then $K = K_\infty \times \prod_{v \text{ finite}} K_v$ is the maximal compact subgroup of $G(\mathfrak{a}_0)$ and $G(\mathfrak{a}_0) = B(\mathfrak{a}_0)K$. The embedding $X(T)_F \hookrightarrow X(T)_F_v$ induces an embedding $\mathfrak{a}_v \hookrightarrow \mathfrak{a}$, where $\mathfrak{a}_v = \text{Hom} (X(T)_F_v, \mathbb{R})$.

There exists a homomorphism $H_B : T(A) \to \mathfrak{a}$, defined by

$$
\exp (\chi, H_B(t)) = \prod_v |\chi(t_v)|_v,
$$

where $\chi \in X(T)_F$ and $t = (t_v)$. We will extend $H_B$ to $G$ by making it trivial on $U$ and $K$. If we define $H_{B_v} : T_v \to \mathfrak{a}_v$ by

$$
q_v^{\langle \chi, H_{B_v}(t) \rangle} = |\chi(t)|_v,
$$

where $\chi \in X(T)_{F_v}$, $t \in T_v$, and $q_v$ is the number of elements in the residue field, when $v$ is finite, and by

$$
\exp (\chi, H_{B_v}(t)) = |\chi(t)|_v,
$$

for $v$ infinite, then

$$
\exp (\chi, H_B(t)) = \prod_{v < \infty} \exp (\chi, H_{B_v}(t_v)) \prod_{v > \infty} q_v^{\langle \chi, H_{B_v}(t_v) \rangle}.
$$

Observe that for almost all $v$, $t_v \in G(O_v)$ on which $H_{B_v}$ is trivial. Thus the product is in fact finite.

**4.1. Definition of Eisenstein series.** For $\mu_1, \ldots, \mu_n$ grösse characters of $F$, we define a character $\chi = \chi(\mu_1, \ldots, \mu_n)$ of $T(\mathfrak{a}_0)$ by

$$
\chi(\mu_1, \ldots, \mu_n)(t(\lambda_1, \ldots, \lambda_n)) = \mu_1(\lambda_1) \cdots \mu_n(\lambda_n).
$$

Let $I(\chi)$ be the space of functions $\Phi$ on $G(\mathfrak{a}_0)$ satisfying $\Phi(utg) = \chi(t)\Phi(g)$ for any $u \in U(\mathfrak{a}_0)$, $t \in T(\mathfrak{a}_0)$ and $g \in G(\mathfrak{a}_0)$. Then for each $\Lambda \in \mathfrak{a}_0^\times$, the representation of $G(\mathfrak{a}_0)$ on the space of functions of the form

$$
g \mapsto \Phi(g) \exp (\Lambda + \rho_B, H_B(g)), \quad \Phi \in I(\chi),
$$

where $\rho_B$ is the half-sum of the positive roots of $B$. This definition, due to G. Shimura, is the one most commonly used in the literature.
is equivalent to \( I(\Lambda, \chi) = \text{Ind}_V \otimes \exp(\Lambda, H_B( )) \). We form the Eisenstein series:

\[
E(g, f, \Lambda) = \sum_{\gamma \in B(F) \setminus G(F)} f(\gamma g),
\]

where \( f = \Phi e^{(\Lambda + \rho_B, H_B( ))} \in I(\Lambda, \chi) \) and \( \rho_B \) is the half-sum of positive roots, i.e., \( \rho_B = \omega_1 + \cdots + \omega_n \). It converges absolutely for \( \text{Re} \Lambda \in \mathbb{C}^+ + \rho_B \) and extends to a meromorphic function of \( \Lambda \). It is an automorphic form and the constant term of \( E(g, f, \Lambda) \) along \( B \) is given by

\[
E_0(g, f, \Lambda) = \int_{U(F) \setminus U(\mathbb{A})} E(ug, f, \Lambda) \, du = \sum_{w \in W} M(w, \Lambda, \chi) f(g),
\]

where \( W \) is the Weyl group of \( T \) and

\[
M(w, \Lambda, \chi) f(g) = \int_{wU(\mathbb{A})w^{-1} \cap U(\mathbb{A}) \setminus U(\mathbb{A})} f(w^{-1}ug) \, du.
\]

Then \( M(w, \Lambda, \chi) \) defines an intertwining map from \( I(\Lambda, \chi) \) to \( I(w\Lambda, w\chi) \) and satisfies a functional equation of the form

\[
M(w_1w_2, \Lambda, \chi) = M(w_1, w_2\Lambda, w_2\chi)M(w_2, \Lambda, \chi).
\]

Let \( S \) be a finite set of places of \( F \), including all the archimedean places such that for every \( v \notin S \), \( \chi_v, \eta_v \) are unramified and if \( f = \otimes f_v \), for \( v \notin S \), \( f_v \) is the unique \( K_v \)-fixed function normalized by \( f_v(e_v) = 1 \). We have

\[
M(w, \Lambda, \chi) = \otimes_v A(w, \Lambda, \chi_v).
\]

Then by applying Gindikin-Karpelevich method, we can see that for \( v \notin S \),

\[
A(w, \Lambda, \chi_v) f_v = \prod_{\alpha > 0, \omega \alpha < 0} \frac{L(\langle \Lambda, \alpha' \rangle, \chi_v \circ \alpha')}{L(\langle \Lambda, \alpha' \rangle + 1, \chi_v \circ \alpha')} f_v,
\]

where \( f_v \) is the \( K_v \)-fixed function in the space of \( I(w\Lambda, w\chi) \) (cf. [6, 17, 18, 27, 28]). For any \( v \), let

\[
r_v(w) = \prod_{\alpha > 0, \omega \alpha < 0} \frac{L(\langle \Lambda, \alpha' \rangle, \chi_v \circ \alpha')}{L(\langle \Lambda, \alpha' \rangle + 1, \chi_v \circ \alpha')L(\langle \Lambda, \alpha' \rangle, \chi_v \circ \alpha', \eta_v)}.
\]

We normalize the intertwining operators \( A(w, \Lambda, \chi_v) \) for all \( v \) by

\[
A(w, \Lambda, \chi_v) = r_v(w) R(w, \Lambda, \chi_v).
\]
Let $R(w, \Lambda, \chi) = \otimes_y R(w, \Lambda, \chi_y)$ and

$$r(w) = \prod_v r_v(w) = \prod_{\alpha > 0, \omega_\alpha < 0} \frac{\xi(\langle \Lambda, \alpha^\vee \rangle, \chi \circ \alpha^\vee)}{\xi(\langle \Lambda, \alpha^\vee \rangle + 1, \chi \circ \alpha^\vee)\epsilon(\langle \Lambda, \alpha^\vee \rangle, \chi \circ \alpha^\vee)}.$$ 

$R(w, \Lambda, \chi)$ satisfies the functional equation

$$R(w_1 w_2, \Lambda, \chi) = R(w_1, w_2 \Lambda, w_2 \chi) R(w_2, \Lambda, \chi),$$

for any $w_1, w_2$. We know, by Winarsky [32] for $p$-adic cases and by Shahidi [26, p. 110] for real and complex cases that

$$A(w, \Lambda, \chi_v) \prod_{\alpha > 0, \omega_\alpha < 0} L_v(\langle \Lambda, \alpha^\vee \rangle, \chi_v \circ \alpha^\vee)^{-1}$$

is holomorphic for any $v$. So for any $v$, $R(w, \Lambda, \chi_v)$ is holomorphic for $\Lambda$ with $\text{Re}(\langle \Lambda, \alpha^\vee \rangle) > -1$, for all positive $\alpha$ with $\omega_\alpha < 0$. For $\chi = \chi(\mu_1, \ldots, \mu_n)$,

$$\chi \circ \alpha^\vee = \begin{cases} \mu_i \mu_j^{-1}, & \text{for } \alpha = e_i - e_j \\ \mu_i \mu_j, & \text{for } \alpha = e_i + e_j \text{ and } i < j \\ \mu_i, & \text{for } \alpha = 2e_i. \end{cases}$$

For $\alpha \in \Phi^+$, let $S_\alpha = \{ \Lambda \in a_\mathbb{C}^n | \langle \Lambda, \alpha^\vee \rangle = 1 \}$. We call $S_\alpha$ a singular hyperplane. We say that $E(g, f, \Lambda)$ has a pole of order $l$ at $\Lambda_0$ if $\Lambda_0$ is the intersection of $l$ singular hyperplanes in general position on which the Eisenstein series has a simple pole.

Langlands’ theory [18, 25] says that $L_\mathbb{A}(B)$ is generated by square integrable iterated residues of $E(g, f, \Lambda)$ at poles of order $n$.

We recall Langlands’ square integrability criterion for automorphic forms through their constant terms in our case ([18, p. 104] or [9, p. 187]). We write the intertwining operator $M(w, \Lambda, \chi)$ as follows:

$$M(w, \Lambda, \chi)f(g) = T(w, \Lambda, \chi) \Phi(g) e^{\langle w \Lambda + p_B, H_B(g) \rangle}.$$ 

Suppose the iterated residue of $E_0(g, f, \Lambda)$ at $\Lambda = \beta$ is given by

$$\text{Res}_\beta E_0(g, f, \Lambda) = \sum_{w \in \Omega} \text{Res}_\beta T(w, \Lambda, \chi) \Phi(g) e^{\langle w \beta + p_B, H_B(g) \rangle}.$$ 

Here $\Omega$ is the set of all $w \in W$ which contribute a nonzero residue. Then we have

**Lemma.** (Langlands) $\text{Res}_\beta E(g, f, \Lambda)$ is square integrable if and only if $\text{Re}(w\beta)$ is in $-\{\sum_{i=1}^{2n} a_i |a_i| a_i > 0\}$ for all $w \in \Omega$. 


For $\Psi \subset \Phi^+$, we define $r(w, \Lambda, \Psi)$ by

$$r(w, \Lambda, \Psi) = \prod_{\alpha \in \Psi, \omega_\alpha < 0} \frac{\xi((\Lambda, \alpha^\vee), \chi \circ \alpha^\vee)}{\xi((\Lambda, \alpha^\vee) + 1, \chi \circ \alpha^\vee)\xi((\Lambda, \alpha^\vee), \chi \circ \alpha^\vee)}.$$ 

Observe that we have suppressed the dependence of $r(w, \Lambda, \Psi)$ on $\chi$.

### 4.2. Residues of the Eisenstein series

We start with

**Proposition 4.1.** Let $E(g, f, \Lambda)$ be the Eisenstein series associated to the trivial character. Its constant term $E_0(g, f, \Lambda)$ is given by

$$E_0(g, f, \Lambda) = \sum_{w \in W} r(w, \Lambda, \Phi^+) R(w, \Lambda, 1)f.$$ 

Let $\Lambda_0 = \rho_B$ be the half-sum of positive roots. Then only $w = w_0$, the longest element of the Weyl group, contributes a pole of order $n$, and the residue of $E(g, f, \Lambda)$ at $\Lambda_0$ is constant.

**Proof:** Note that $\{\alpha | (\rho_B, \alpha^\vee) = 1\}$ is the set of simple roots. Therefore, $\rho_B$ is the intersection of the $n$ singular hyperplanes $S_\alpha$ for simple roots $\alpha$. But

$$\{w | w\alpha < 0, \text{for all simple roots } \alpha\} = \{w_0, \text{the longest Weyl group elements in } W\}.$$ 

Therefore, the residue at $\lambda = \rho_B$ is

$$(*) \otimes_V R_V(w_0, \rho_B, \chi_V) f_V,$$

where $f_V \in I_V(\rho_B, \chi_V)$. But $R_V(w_0, \rho_B, \chi_V) I_V(\rho_B, \chi_V)$ is the Langlands’ quotient, which is constant. Therefore, the residue is constant.

**Remark 1.** Here the half-sum of positive roots corresponds, by (2.1), to the unipotent orbits with Jordan blocks $(2n + 1)$ for $G^* = O_{2n+1}(\mathbb{C})$, $(2n - 1, 1)$ for $G^* = O_{2n}(\mathbb{C})$, resp. i.e., $\rho_B = \xi_1 e_1 + \frac{\xi_2}{2} e_2 + \cdots$. We note that $\Lambda_0$ and $w_0$ satisfy $(\Lambda_0, e_n) = 0$ and $w_0\Lambda_0 = -\Lambda_0$, the first only valid for $G = SO_{2n}$.

For $\chi$ a nontrivial character, we can assume, after conjugation, that $\chi = \chi(\mu_1, \ldots, \mu_1, \ldots, \mu_k, \ldots, \mu_k, 1, \ldots, 1)$, $r_0 + \cdots + r_k = n$, $r_1 \geq \cdots \geq r_k$.

Let $E(g, f, \Lambda)$ be the Eisenstein series associated to the character $\chi$.

**Proposition 4.2.** The Eisenstein series has a pole of order $n$ only if $r_k \geq 2$ and $\mu_i$ is a quadratic gr"ossencharakter for $i = 1, \ldots, k$. 
We divide the set of positive roots $\Phi^+$ as follows:

$\Phi_1 = \{e_i \pm e_j, \ 1 \leq i < j \leq r_1\}$,

$\Phi_2 = \{e_{r_1+i} \pm e_{r_1+j}, \ 1 \leq i < j \leq r_2\}$,

$\vdots$

$\Phi_k = \{e_{r_1+\ldots+r_{k-1}+i} \pm e_{r_1+\ldots+r_{k-1}+j}, \ 1 \leq i < j \leq r_k\}$,

$\Phi_0 = \{e_{r_1+\ldots+r_k+i} \pm e_{r_1+\ldots+r_k+j}, \ 1 \leq i < j \leq r_0, \ 2e_{r_1+\ldots+r_k+i}, \ i = 1, \ldots, r_0\}$,

$\Phi_D = \Phi^+ - \bigcup_{i=0}^k \Phi_i$.

$\Phi_1, \ldots, \Phi_k$ are root systems of type $D_n$ and $\Phi_0$ is a root system of type $C_n$. This corresponds to the decomposition $O(V_1) \times \cdots \times O(V_k) \times O(V_0) \subset O_{2n+1}(\mathbb{C})$. Let $W_i$ be the Weyl group corresponding to $\Phi_i$ for $i=0,\ldots,k$. Let $\Lambda = \Lambda_1 + \cdots + \Lambda_k + \Lambda_0$, where $\Lambda_i = a_{r_1+\ldots+r_{i-1}+1}e_{r_1+\ldots+r_{i-1}+1} + \cdots + a_{r_1+\ldots+r_{i}}e_{r_1+\ldots+r_{i}}$ for $i = 1, \ldots, k$ and $\Lambda_0 = a_{r_1+\ldots+r_k+1}e_{r_1+\ldots+r_k+1} + \cdots + a_ne_n$.

We recall the following well-known result (Carter [5, p. 47]).

**Proposition 4.3.** Let $\Delta$ be a set of simple roots and $W$ be the associated Weyl group. Let $w_\alpha$ be the simple reflexion with respect to $\alpha \in \Delta$. Then $W$ is generated by the $w_\alpha$, $\alpha \in \Delta$. Let $\theta$ be a subset of $\Delta$ and $W_\theta$ be the subgroup of $W$ generated by the $w_\alpha$, $\alpha \in \theta$. Then each coset $ww_\theta$ has a unique element $d_\theta$ characterized by any of the following equivalent properties:

1. $d_\theta \theta > 0$;
2. $d_\theta$ is of minimal length in $ww_\theta$; and
3. For any $x \in W_\theta$, $l(d_\theta x) = l(d_\theta) + l(x)$.

We apply Proposition 4.3 to $\Delta = \{e_1 - e_2, \ldots, e_{n-1} - e_n\}$ and $\theta = \Delta - \{e_{r_1} - e_{r_1+1}, e_{r_1+2} - e_{r_1+2+1}, \ldots, e_{r_1+\ldots+r_k} - e_{r_1+\ldots+r_k+1}\}$. Let $D$ be the set of such distinguished coset representatives. Then we have

**Proposition 4.4.** The constant term $E_0(g, f, \Lambda) = \sum_{w \in W} r(w, \Lambda, \Phi^+) R(w, \Lambda, \chi)f$ can be written as

$$\prod_{i=1}^k \sum_{w_i \in W_i} r(w_i, \Lambda_i, \Phi_i) \sum_{w_0 \in W_0} r(w_0, \Lambda_0, \Phi_0) \cdot \sum_{d \in D} \sum_{c \in C} r(dcw_1 \cdots w_kw_0, \Phi_D) R(dcw_1 \cdots w_kw_0, \Lambda, \chi)f,$$

where $C$ is the set spanned by $c_{r_1}, c_{r_1+r_2}, \ldots, c_{r_1+\ldots+r_k}$. Here $c_i$'s are sign changes in the Weyl group: its action on $\Phi(G, T)$ takes $e_i$ to $-e_i$.

Let $\Lambda_0 = \Lambda_{1,0} + \cdots + \Lambda_{k,0} + \Lambda_{0,0}$, where $\Lambda_{i,0}$ is the half-sum of positive roots in $\Phi_i$ for $i = 0, \ldots, k$. Then
Theorem 4.5. The residue of $E_0(g,f,\Lambda)$ at $\Lambda_0$ is given by

\begin{equation}
\sum_{d \in D} (\ast)R(dw_0,\Lambda_0,\chi) \prod_{i=1}^k (1 + R(c_{r_1+\ldots+r_i}))f,
\end{equation}

where $w_0$ is determined by $\Lambda_0$ and $(\ast)$ signifies a constant. It is square integrable.

It is instructive to consider first a simple case to illustrate our method, namely, $\chi = \chi(\mu, \ldots, \mu)$, $\mu$ nontrivial and quadratic. The main idea of our proof is already contained in this simple case.

Let $\Phi_1 = \{e_i \pm e_j, \ 1 \leq i < j \leq n\}$ and $\Phi_2 = \{2e_i, \ i = 1, \ldots, n\}$. Then for $\alpha \in \Phi_1$, $\chi \circ \alpha^\vee = 1$ and for $\alpha \in \Phi_2$, $\chi \circ \alpha^\vee$ is nontrivial. Let $W_1$ be the Weyl group associated to $\Phi_1$. It is a Weyl group of type $D_n$. Here $(W/W_1) = 2$ and the nontrivial coset has a distinguished coset representative, i.e., $c_n$. It is the unique element which satisfies $c_n\Phi_1 > 0$. Here for $w \in W_1$,

$$\{\alpha > 0 \ | \ c_n\alpha < 0\} = \{\alpha \in \Phi_1 \ | \ w\alpha < 0\} \cup \{\alpha \in \Phi_2 \ | \ c_n\alpha < 0\}.$$ 

Therefore the constant term of the Eisenstein series is

$$\sum_{w_1 \in W_1} r(w_1,\Lambda,\Phi_1)(r(w_1,\Lambda,\Phi_2)R(w_1,\Lambda,\chi) + r(c_nw_1,\Lambda,\Phi_2)R(c_nw_1,\Lambda,\chi)).$$

We consider the residue at $\Lambda = \Lambda_0$, the half-sum of positive roots of $\Phi_1$. Since $(\Lambda_0,e_n) = 0$, the last term is holomorphic on every singular hyperplane. The first term has a pole of order $n$ at $w = w_0$, the longest element of $W_1$. Since $w_0\Lambda_0 = -\Lambda_0$, $w_0e_i = \pm e_k$, for some $k < n$ depending on $i$. Therefore, for $\alpha = 2e_i$, $i = 1, \ldots, n - 1$, $w_0\alpha < 0$ if and only if $c_nw_0\alpha < 0$. So $r(w_0,\Lambda_0,\Phi_2) = r(c_nw_0,\Lambda_0,\Phi_2)$, and the residue is

$$(\ast)R(c_nw_0,\Lambda_0,\chi)(1 + R(c_n,\Lambda_0,\chi)),$$

since $c_nw_0c_n = w_0$.

4.3. Proof of Proposition 4.2. We need

Proposition 4.6. If one of $\mu$ is not a quadratic grössencharacter or $r_i = 1$ for some $i > 0$, then the Eisenstein series has no pole of order $n$. In particular, for $\chi = \chi(\mu, \ldots, \mu, \nu_1, \ldots, \nu_{r_1})$ (if $r_1 > 1$, $\mu$ is not quadratic), the Eisenstein series has no pole of order $n$. 
Proof. Let
\[
\Phi_1 = \{e_i - e_j, \ 1 \leq i < j \leq r_1\},
\]
\[
\Phi_2 = \{e_{r_1+i} \pm e_{r_1+j}, \ 1 \leq i < j \leq r_2, \ 2e_{r_1+i}, \ i = 1, \ldots, r_2\},
\]
\[
\Phi_3 = \Phi^+ - \Phi_1 \cup \Phi_2.
\]
Then \(\Phi_1\) is a root system of type \(A_{r_1-1}\) and \(\Phi_2\), a root system of type \(C_{r_2}\). For \(\alpha \in \Phi_3\), \(\chi \circ \alpha^\vee\) is nontrivial. Let \(W_i\) be the Weyl group of \(\Phi_i\) for \(i = 1, 2\). Let \(\Lambda = \Lambda_1 + \Lambda_2\), where \(\Lambda_1 = a_1e_1 + \cdots + a_r e_r\) and \(\Lambda_2 = a_{r+1}e_{r+1} + \cdots + a_n e_n\). Then \(W_1W_2\) is \(W_\theta\) in Proposition 3 with \(\theta = \Delta - \{e_1 - e_{r+1}\}\). Let \(D\) be the set of distinguished coset representatives for \(W/W_1W_2\). For \(d \in D, w_1 \in W_1\) and \(w_2 \in W_2\),
\[
\{ \alpha > 0: dw_1w_2 \alpha < 0 \} = \{ \alpha \in \Phi_1 | w_1 \alpha < 0 \}
\]
\[
\cup \{ \alpha \in \Phi_2 | w_2 \alpha < 0 \}
\]
\[
\cup \{ \alpha \in \Phi_3 | dw_1w_2 \alpha < 0 \}.
\]
Using (4.1), we write the constant term of the Eisenstein series
\[
E_0(g, f, \Lambda) = \sum_{w \in W} r(w, \Lambda, \Phi^+) R(w, \Lambda, \chi)f_w
\]
as follows:
\[
E_0(g, f, \Lambda) = \sum_{w \in W} \tilde{r}(w, \Lambda, \Phi^+) \otimes_{\mathfrak{g} \neq S} \tilde{f}_w \otimes \otimes_{v \in S} \tilde{A}(w, \Lambda, \chi)f_v,
\]
where \(\tilde{A}(w, \Lambda, \chi)\) is the expression (4.1) and
\[
\tilde{r}(w, \Lambda, \Phi^+) = \prod_{\alpha \in \Phi^+, \alpha \alpha < 0} \frac{\xi_3((\Lambda, \alpha^\vee), \chi)}{\xi_3((\Lambda, \alpha^\vee) + 1, \chi)},
\]
where \(\xi_3(z, \chi) = \prod_{v \not\in S} L(z, \chi v)\) is the partial Hecke \(L\)-function. Then we have
\[
E_0(g, f, \Lambda) = \sum_{w_1 \in W_1} \tilde{r}(w_1, \Lambda_1, \Phi_1) \sum_{w_2 \in W_2} \tilde{r}(w_2, \Lambda_2, \Phi_2)
\]
\[
\sum_{d \in D} \tilde{r}(dw_1w_2, \Lambda, \Phi_3) \otimes_{\mathfrak{g} \neq S} \tilde{f}_v \otimes \otimes_{v \in S} \tilde{A}(dw_1w_2, \Lambda, \chi)f_v.
\]
\(\tilde{A}(w, \Lambda, \chi)\) is entire and \(\tilde{r}(w, \Lambda, \Phi_3)\) is holomorphic on any singular hyperplane. Any pole of the first term is an intersection of \(\leq r_1 - 1\) singular hyperplanes in general position and any pole of the second term is an intersection of \(\leq r_2\) singular hyperplanes in general position. Therefore, any pole of the Eisenstein
series is an intersection of \( \leq r_1 + r_2 - 1 < n \) singular hyperplanes in general position. This proves Proposition 4.6.

Proposition 4.2 is now immediate.

4.4. Proof of Proposition 4.4. First we prove the assertion for

\[
\chi = \chi(\mu, \ldots, \mu, \nu_1, \ldots, \nu_{n-r_1}),
\]

where \( \mu \) is nontrivial, quadratic and \( \mu \) and \( \nu_j \) are distinct for all \( j \).

Let \( \chi_1 = \chi(\mu, \ldots, \mu) \) and \( \chi_2 = \chi(\nu_1, \ldots, \nu_{n-r_1}) \). Let

\[
\Phi_1 = \{ e_i \pm e_j \mid 1 \leq i < j \leq r_1 \},
\]

\[
\Phi_2 = \{ e_{r_1+i} \pm e_{r_1+j}, \quad 1 \leq i < j \leq n - r_1, \quad 2e_{r_1+i}, i = 1, \ldots, n - r_1 \},
\]

\[
\Phi_3 = \Phi - \Phi_1 \cup \Phi_2 = \{ e_i \pm e_{r_1+j}, \quad i = 1, \ldots, r_1, j = 1, \ldots, n - r_1, \quad 2e_i, i = 1, \ldots, r_1 \}.
\]

Then for \( \alpha \in \Phi_1, \chi \circ \alpha^\vee = 1 \). For \( \alpha \in \Phi_2, \chi \circ \alpha^\vee = \chi_2 \circ \alpha^\vee \) and for \( \alpha \in \Phi_3, \chi \circ \alpha^\vee \) is nontrivial. Let \( W_i \) be the Weyl group associated to \( \Phi_i, i = 1, 2 \). Let \( \Lambda = \Lambda_1 + \Lambda_2, \Lambda_1 = a_1 e_1 + \cdots + a_1 e_{r_1} \) and \( \Lambda_2 = a_{r_1+1} e_{r_1+1} + \cdots + a_n e_n \). We apply Proposition 4.3 to \( \Delta = \{ e_1 - e_2, \ldots, e_{n-1} - e_n \} \) and \( \theta = \Delta - \{ e_{r_1} - e_{r_1+1} \} \). Let \( D_\theta \) be the set of distinguished coset representatives. Then we need

**Lemma 4.7.** \( D = D_\theta \cup D_{\theta c_{r_1}} \) is the set of distinguished coset representatives for \( W/W_1 W_2 \), i.e., \( d \in D \) if and only if \( d(\Phi_1 \cup \Phi_2) > 0 \).

**Proof.** Since \( D_\theta \) contains no sign changes, it follows immediately that \( d(\Phi_1 \cup \Phi_2) > 0 \) for all \( d \in D \). It can be easily checked that \( \#D = \#(W/W_1 W_2) \). We therefore only need to show that each coset has a unique coset representative in \( D \). Suppose \( d_i \in D \) for \( i = 1, 2 \) and \( d_1^{-1} d_2 = w_1 w_2 \in W_1 W_2 \). Then \( d_1 = d_1 w_1 w_2 \).

Using \( d_2 \alpha > 0 \) for all \( \alpha \in \Phi_2 \) implies that \( d_1 w_1 w_2 \alpha > 0 \). Here \( w_1 \) and \( w_2 \) commute and \( w_1 \alpha = \alpha \) for \( \alpha \in \Phi_2 \). Therefore, we have \( d_1 w_2 \alpha > 0 \) for all \( \alpha \in \Phi_2 \) or \( w_2 \alpha > 0 \) for all \( \alpha \in \Phi_2 \). This implies that \( w_2 = 1 \). In the same way, we have \( w_1 = 1 \). This proves the lemma.

For \( d \in D \), \( w_1 \in W_1 \) and \( w_2 \in W_2 \),

\[
\{ \alpha > 0: dw_1 w_2 \alpha < 0 \} = \{ \alpha \in \Phi_1 \mid w_1 \alpha < 0 \}
\]

\[
\cup \{ \alpha \in \Phi_2 \mid w_2 \alpha < 0 \}
\]

\[
\cup \{ \alpha \in \Phi_3 \mid dw_1 w_2 \alpha < 0 \}.
\]
Then the constant term of $E(g,f,\Lambda)$ is given by

\[
(4.3) \sum_{w_2 \in W_2} r(w_2,\Lambda,\Phi_1) \sum_{w_2 \in W_2} r(w_2,\Lambda,\Phi_2) \cdot \sum_{d \in D_\theta} (r(dw_1w_2,\Lambda,\Phi_3)R(dw_1w_2,\Lambda,\chi) + r(dc_r_1w_1w_2,\Lambda,\Phi_3)R(dc_r_1w_1w_2,\Lambda,\chi)).
\]

In order to apply induction, let $\chi = \chi(\mu_1,\ldots,\mu_2,\ldots,\nu_1,\ldots,\nu_l, r_1 + r_2 + l = n)$. Let $\chi_2 = \chi(\mu_2,\ldots,\mu_2,\nu_1,\ldots,\nu_l)$. We repeat the above for $\chi_2$ and divide $\Phi_2$ as follows:

\[
\begin{align*}
\Phi_4 &= \{e_{r_{i+1}} \pm e_{r_{i+j}}, \quad 1 \leq i < j \leq r_2\}, \\
\Phi_5 &= \{e_{r_{i+1+j}} \pm e_{r_{i+j+1}}, \quad 1 \leq i < j \leq l, \quad 2e_{r_{i+r_2+j}}, \quad i = 1, \ldots, \lfloor l/2 \rfloor\}, \\
\Phi_6 &= \Phi_2 - \Phi_4 \cup \Phi_5.
\end{align*}
\]

Let $W_i$ be the Weyl group of $\Phi_i$ for $i = 4,5$. Then $D' = D_\theta \cup D_\theta c_{r_1+r_2}$ is the set of distinguished coset representatives for $W_2/W_4 W_5$, where $D_\theta$ is the set of distinguished coset representatives for

\[
\theta' = \{e_{r_{i+1}} - e_{r_{i+2}}, \ldots, e_{r_{n-1}} - e_{r_n}\} - \{e_{r_{1+r_2}} - e_{r_{1+r_2+1}}\}.
\]

Then one can show that $D_\theta D_\theta$ is the set of distinguished coset representatives in Proposition 4.3 for $\{e_1 - e_2, \ldots, e_{n-1} - e_n\} - \{e_{r_1 - e_{r_1+1}}, e_{r_1 + r_2} - e_{r_1 + r_2+1}\}$ and

\[
DD' = D_\theta D_\theta \cup D_\theta D_\theta c_{r_1} \cup D_\theta D_\theta c_{r_1 + r_2} \cup D_\theta D_\theta c_{r_1 + r_2 + r}
\]

is the set of distinguished coset representatives for $W/W_1 W_4 W_5$, i.e., $d \in DD'$ if and only if $d(\Phi_1 \cup \Phi_4 \cup \Phi_5) > 0$. Proposition 4.4 now follows by induction.

**4.5. Proof of Theorem 4.5.** We apply induction and start with the equation (4.3). Suppose the first term has a pole at $\Lambda_{1,0}$ and $w_1 = w_{\Lambda_{1,0}}$ contributes the pole. Let $\Lambda = \Lambda_{1,0} + \Lambda_2$. We need

**Lemma 4.8.** For each $w_2 \in W_2$,

\[
D(w_{\Lambda_{1,0}} w_2,\Lambda,\Phi_3) = r(pc_r_1 w_{\Lambda_{1,0}} w_2,\Lambda,\Phi_3).
\]

**Proof.** Recall the properties of $w_1$ and $\Lambda_{1,0}$: $(\Lambda_{1,0}, e_{r_1}) = 0$ and $w_1 \Lambda_{1,0} = -\Lambda_{1,0}$.

Therefore, $\langle \Lambda, \alpha \rangle = 0$ for $\alpha = 2e_{r_1}$. For $i < r_1$, $w_1 e_i = \pm e_k$, $k < r_1$ and
\[ w_{1}e_{r_{1}} = \pm e_{r_{1}}. \] So for \( \alpha = e_{i} \pm e_{r_{i}+j}, 2e_{i}, i = 1, \ldots, r_{1} - 1, j = 1, \ldots, n - r_{1}, \]
\[ dw_{1}w_{2} < 0 \] if and only if \( dc_{r_{1}}w_{1}w_{2} \alpha < 0 \) since \( w_{1}w_{2} = e_{r_{1}}w_{1}w_{2} \alpha. \)

For \( \alpha = e_{r_{1}} \pm e_{r_{1}+j}, \) we have

**Lemma 4.9.** Only one of the following is possible: either \( \alpha \) satisfies both \( dw_{1}w_{2} \alpha < 0 \) and \( dc_{r_{1}}w_{1}w_{2} \alpha < 0, \) or \( \alpha \) and \( e_{r_{1}} \pm e_{r_{1}+j} \) satisfy only one inequality.

**Proof.** Suppose \( dw_{1}w_{2}(e_{r_{1}} \pm e_{r_{1}+j}) = d(-e_{r_{1}} \mp e_{r_{1}+k}). \) Then \( dc_{r_{1}}w_{1}w_{2}(e_{r_{1}} \pm e_{r_{1}+j}) = d(e_{r_{1}} \mp e_{r_{1}+k}). \) Since \( d \) is a permutation, we have our assertion. The other case is similar, completing the lemma.

If \( \alpha = e_{r_{1}} \pm e_{r_{1}+j} \) satisfy \( dw_{1}w_{2} \alpha < 0, \) then \( \langle \Lambda, \alpha^\vee \rangle = \pm \langle \Lambda_{2}, e_{r_{1}+j} \rangle \) and

\[
\frac{\xi((\Lambda_{2}, e_{r_{1}+j}), \mu \nu_{j})}{\xi((\Lambda_{2}, e_{r_{1}+j}) + 1, \mu \nu_{j} \epsilon((\Lambda_{2}, e_{r_{1}+j}), \mu \nu_{j})}
\times \frac{\xi(- (\Lambda_{2}, e_{r_{1}+j}), \mu \nu_{j})}{\xi(-(\Lambda_{2}, e_{r_{1}+j}) + 1, \mu \nu_{j} \epsilon(-(\Lambda_{2}, e_{r_{1}+j}), \mu \nu_{j})} = 1,
\]

using the functional equation \( \xi(z, \mu) = \epsilon(z, \mu) \xi(1-z, \mu) \) for \( \mu \) a nontrivial quadratic grössencharacter.

This proves Lemma 4.8.

It now follows that the residue at \( \Lambda, \) as a function of \( \Lambda_{2}, \) is

\[
\sum_{w_{2} \in W_{2}} r(w_{2}, \Lambda_{2}, \Phi_{2})(\sum_{d \in D} (*) R(dw_{2}w_{1,0}, \Lambda, \chi)(1 + R(c_{r_{1}})),
\]

where \( w_{1,0} = c_{r_{1}}w_{A_{0},0} \) since \( c_{r_{1}}w_{A_{0},0}c_{r_{1}} = w_{A_{1},0}. \) Theorem 4.5 now follows by applying induction. We only need:

**Proposition 4.10.** The residue in Theorem 4.5 is square integrable.

**Proof.** By Langlands' Lemma and the fact that \( w_{0}\Lambda_{0} = -\Lambda_{0}, \) it is enough to show that \( d\Lambda_{0} \) is a linear combination of simple roots with positive coefficients for any \( d \in D. \)

First of all, it is easy to see that any linear combination of \( e_{i} \)'s with non-negative coefficients which contains \( e_{1} \) is a linear combination of simple roots with positive coefficients. \( \Lambda_{0} \) satisfies this property. Since \( d \) is a permutation, it is easy to show that \( d\Lambda_{0} \) contains \( e_{1}. \)

Recall the property of \( \Lambda_{0} \) that \( \Lambda_{0} \) contains \( e_{1}, e_{r_{1}+1}, \ldots, e_{r_{1}+r_{2}+\ldots+r_{k}+1}. \) Also recall the property of \( d \in D \) from Proposition 4.3: \( d\theta > 0 \) where \( \theta = \{ e_{1} - e_{2}, \ldots, e_{r_{1}+1} - e_{r_{1}} \} \cup \{ e_{r_{1}+1} - e_{r_{1}+2}, \ldots, e_{r_{1}+r_{2}+1} - e_{r_{1}+r_{2}} \} \cup \ldots \cup \{ e_{r_{1}+r_{2}+\ldots+r_{k}+1} - e_{r_{1}+r_{2}+\ldots+r_{k}+2}, \ldots, e_{r-1} - e_{r} \}. \) Hence, one of \( e_{1}, e_{r_{1}+1}, \ldots, e_{r_{1}+r_{2}+\ldots+r_{k}+1} \) is sent to \( e_{1} \) by \( d \in D. \) So \( d\Lambda_{0} \) contains \( e_{1}. \)
5. Arthur parameters for the residual spectrum. In this section we interpret Theorem 4.5 in terms of Arthur parameters. Recall the quadratic Arthur parameters in our case: we have a decomposition \( \mathbb{C}^{2n+1} = V_0 \oplus V_1 \oplus \cdots \oplus V_k \), where \( \dim V_0 = 2r_0 + 1 \), \( \dim V_i = 2r_i \) for \( i = 1, \ldots, k \), \( r_1 \geq r_2 \geq \cdots \geq r_k \geq 2 \), \( r_0 + \cdots + r_k = n \), and embedding \( \prod_{i=0}^k O(V_i) \subset O_{2n+1}(\mathbb{C}) \). Let \( \mu_1, \ldots, \mu_k \) be distinct quadratic grössencharacters and \( O_i \) be the unipotent orbits with Jordan blocks \((2r_i - 1, 1)\) for \( i = 1, \ldots, k \) and \((2r_0 + 1)\) for \( i = 0 \) (see the remark after Proposition 4.1). Then the Arthur parameter of interest to us is the homomorphism

\[
\psi: W_F \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \longrightarrow \prod_{i=0}^k O(V_i) \subset O_{2n+1}(\mathbb{C}),
\]

satisfying:

1. \( \psi|_{W_F}: w \longmapsto 1 \times \mu_1(w) \times \cdots \times \mu_k(w) \in \{ \pm 1 \} \times \{ \pm 1 \} \times \cdots \times \{ \pm 1 \} \), where \( \{ \pm 1 \} \) is the center of \( O(V_i) \) for \( i = 0, \ldots, k \);
2. \( \psi|_{SL_2(\mathbb{C}) \times SL_2(\mathbb{C})}: 1 \equiv 1 \); and
3. (3) by Jacobson-Morozov theorem, \( \psi|_{1 \times 1 \times SL_2(\mathbb{C})} \) defines the unipotent orbit \( \prod_{i=0}^k O_i \) of \( G^* \).

Recall that we are considering the residue of the Eisenstein series at \( \Lambda_0 = \Lambda_{1,0} + \ldots + \Lambda_{k,0} + \Lambda_{0,0} \), where each \( \Lambda_{i,0} \) is the half-sum of (positive) roots in \( \Phi_{i,i} \) for all \( i = 0, 1, \ldots, k \). The character \( \chi \) and the quasicharacter \( \exp(\langle \Lambda_0, H_B(\cdot) \rangle) \) of \( T \) may be viewed as homomorphisms from \( W_F \) into \( L^T \) (cf. [14, 16]). The unipotent orbits \( O_i \) are determined by \( \Lambda_0 \) through Jacobson-Morozov’s theorem. Then the associated Langlands’ parameter \( \phi_\psi \), i.e., the homomorphism

\[
\phi_\psi: W_F \times SL_2(\mathbb{C}) \to O_{2n+1}(\mathbb{C})
\]

defined by \( \phi_\psi|_{SL_2(\mathbb{C})} = 1 \) and

\[
\phi_\psi(w) = \psi \left( w, 1, \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix} \right),
\]

is \( \phi_\psi = \chi \otimes \exp(\langle \Lambda_0, H_B(\cdot) \rangle) \) (cf. [1]). Its nontempered part is \( \phi^*_\psi = \exp(\langle \Lambda_0, H_B(\cdot) \rangle) \).

Let \( M^* = \text{Cent} (im\phi^*_\psi, G^*) \). Since \( \langle \Lambda_0, e_i \rangle = 0 \) for \( i = r_1, r_1 + r_2, \ldots, r_1 + \ldots + r_k \), the Levi subgroup \( M \) which has \( M^* \) as its \( L \)-group, will be, up to isomorphism, of the form \( GL_{n_1} \times \ldots \times GL_{n_r} \times \text{Sp}_{2k} \), where \( n_1, \ldots, n_r \) are determined by \( \Lambda_0 \).

The parameter \( \Lambda_0 \) may not be in the positive Weyl chamber of the split component of \( M \). But one can choose an element \( w' \) in the Weyl group of shortest length so that \( \lambda_0 = w'\lambda_0 \) belongs there. Then, using the functional equation, the Eisenstein series attached to \( \lambda \) and \( \chi' = w'\chi \) will have a pole of order \( n \) at \( \lambda = \lambda_0 \). The Arthur parameter which is determined only up to conjugacy will not change. From now on we shall assume that \( \Lambda_0 \) is in the positive Weyl chamber of the split component of \( M \).
For each place \( v \), decompose \( \phi_{\psi_v} \) as \( \phi_{\psi_v} = \phi_{\psi_v}^0 \cdot \phi_{\psi_v}^+ \) as in \([1]\). The parameter \( \phi_{\psi_v}^0 \) factors through \( M^* \) and is the Langlands parameter for the (tempered) constituents of the unitary principal series \( I_v = \text{Ind}_{B_0(F_v)}^{M(F_v)} \chi_v = \oplus_{\tau_v \in I_v} \tau_v \), of \( M(F_v) \), where \( B_0 = B \cap M \). For each \( \pi_{\psi_v} \), let \( \Pi_{\psi_{v, i}} = \text{Ind}_{G_0(F_v)}^{G(F_v)}(\text{res}(\Lambda_0, H_{\text{re}}(\cdot))) \) be the corresponding \( K \)-type \([19]\), where \( P = MN \). Then for each \( v \) the \( L \)-packet parametrized by \( \phi_{\psi_v} \) is \( \Pi_{\phi_{\psi_v}} = \{ \Pi_{\psi_{v, i}} \} \) (cf. \([1, 19]\)). The \( R \)-group for the parameter \( \phi_{\psi_v} \), i.e., \( C_{\phi_{\psi_v}} \) is the same as the \( R \)-group of \( I_v \) for each \( v \) in the sense of Knapp-Stein (cf. \([7, 10, 11, 13]\)).

By theorem \( C_n \) of \([10]\), the \( R \)-group \( C_{\phi_{\psi_v}} \) of \( I_v \) is a subgroup of the group generated by the sign changes \( c_i, i = r_1, r_1 + r_2, r_1 + \ldots + r_k \), a product of 2-groups. Moreover, if the sign change \( c_{r_1 + \ldots + r_i} \) in \((4.2)\) does not belong to \( C_{\phi_{\psi_v}} \) for some \( i \), then the normalized operator \( R(c_{r_1 + \ldots + r_i}) \) acts like identity.

Let \( \pi(\chi_v) = \{ \pi_{\psi_v} \} \). Then, given a place \( v \), in \([11]\) Keys and Shahidi defined a pairing \( \langle \cdot, \cdot \rangle \) on \( C_{\phi_{\psi_v}} \times \pi(\chi_v) \). We extend the pairing \( \langle \cdot, \cdot \rangle \) to \( C_{\phi_{\psi_v}} \times \Pi_{\phi_{\psi_v}} \) as in Arthur \([1, p. 9]\) by setting \( \langle \tau_v, \Pi_{\psi_{v, i}} \rangle = \langle \tau_v, \pi_{\psi_{v, i}} \rangle \). This can further be extended to \( C_{\psi_v} \times \Pi_{\phi_{\psi_v}} \), using the surjection \( C_{\psi_v} \rightarrow C_{\phi_{\psi_v}} \) for each \( v \) \((1, p. 11)\). Let \( \Pi = \otimes_v \Pi_{\psi_{v, i}} \) where almost all \( \Pi_{\psi_{v, i}} \) are spherical. Then \( \Pi 
(\text{global} \) \(-\text{packet of } \phi_{\psi_v} \). Finally set \( \langle \tau, \Pi \rangle = \prod_v \langle \tau_v, \Pi_{\psi_{v, i}} \rangle \), where \( \tau_v \) is the image of \( \tau \) under the map \( C_{\psi_v} \rightarrow C_{\phi_{\psi_v}} \). We need to be more precise since this is an infinite product. For each place \( v \), the corresponding pairing in \([11]\) is defined by means of a nontrivial additive character of \( F_v \). Fix a nontrivial additive character \( \eta = \otimes_v \eta_v \) of \( \mathbb{A}_F \). At each place \( v \), there is a unique representation \( \pi_{v, 0} \in \{ \pi_{\psi_v} \} \) which is generic with respect to \( \eta_v \) and for which \( \langle \tau_v, \pi_{v, 0} \rangle = 1 \) for all \( \tau_v \in C_{\phi_{\psi_v}} \). If \( \eta_v \) is unramified and \( \{ \pi_{v, i} \} \) contains a spherical representation (which is equivalent to \( \chi_v \) being unramified), then it is \( \pi_{v, 0} \). Consequently the product \( \prod_v \langle \tau_v, \Pi_{v, i} \rangle \) is a finite product. Moreover \( \langle \cdot, \Pi \rangle \) does not depend on the choice of \( \eta \).

We should finally mention that by theorem 5.1 of \([11]\) the action of each normalized operator \( R(\tau) \) in \((4.2)\) on a component \( \Pi \) is according to the pairing \( \langle \tau, \Pi \rangle \). (The rank one characters coming from \( \chi \) in the normalized operators \( R(c_i), i = r_1, r_1 + r_2, \ldots, r_1 + \ldots + r_k \), are all nontrivial and therefore the global sign of theorem 5.1 of \([11]\) is 1 for all of them.)

Applying \((4.2)\) to \( \Pi = \otimes_v \Pi_{v, i} \in \Pi_{\phi_{\psi_v}} \) now implies that the residue is equal to

\[
(5.1) \sum_{d \in D} \langle *, R(\delta w_0, \Lambda_0, \chi) \rangle \sum_{x \in C_{\phi_{\psi_v}}} \langle x, \Pi \rangle.
\]

It is now clear that since \( C_{\phi_{\psi_v}} \) is abelian, \((5.1)\) is nonzero if and only if \( \langle *, \Pi \rangle \) is the trivial character. We can therefore reformulate Theorem 4.5 as:

**Theorem 5.1.** \( \Pi \) appears in \( L^2(G(F)\backslash G(\mathbb{A}_F)) \) if and only if \( \langle *, \Pi \rangle \) is the trivial character, i.e., the Arthur condition (cf. \([23]\)) holds.
Since \( C_\psi \) is abelian, this is equivalent to the fact that there exists a positive integer \( d_\psi \) such that \( \Pi \in \Pi_{\phi_\psi} \) appears with multiplicity
\[
\frac{d_\psi}{|C_\psi|} \sum_{\alpha \in C_\psi} \langle x, \Pi \rangle = \frac{d_\psi}{|C_{\phi_\psi}|} \sum_{\alpha \in C_{\phi_\psi}} \langle x, \Pi \rangle.
\]
This proves the global Arthur conjecture on the multiplicity formula for the residual spectrum.

Remark 2. We remark that, in the sum (5.1), \( w_0 \Lambda_0 = \Lambda_0 \) and \( d \Lambda_0 = \Lambda_0 \) imply \( d = 1 \). Therefore there is no cancellation among the summands and the residue is a sum of isomorphic images of the Langlands’ quotient.

5.1. Special case of \( G = \text{Sp}_4 \). The dual group \( G^* = O_5(\mathbb{C}) \) has only one distinguished orbit, namely the principal one given by the Jordan block (5). The corresponding unipotent parameter then parametrizes the constants, the only spherical residue of \( \text{Sp}_4 \) (cf. [22]). To parametrize the rest of the residual spectrum, we construct quadratic unipotent Arthur parameters in the sense of Moeglin as follows: There is a natural embedding of \( O_4(\mathbb{C}) \) into \( O_5(\mathbb{C}) \) by sending
\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in O_4(\mathbb{C}) \text{ into } \begin{pmatrix} A & 0 & B \\ 0 & 1 & 0 \\ C & 0 & D \end{pmatrix} \text{ in } O_5(\mathbb{C}).
\]

For \( \mu \) a nontrivial quadratic gröschencharacter of \( F \), we define \( \psi \) as follows:
1. \( \psi|_{W_F}: w \mapsto \mu(w) \in \{ \pm 1 \} = \text{Center of } O_4(\mathbb{C}); \)
2. \( \psi|_{1 \times SL_2(\mathbb{C}) \times 1} \equiv 1 \); and
3. \( \psi|_{1 \times 1 \times SL_2(\mathbb{C})} \) determines a unipotent orbit with Jordan blocks \( (3,1) \) in \( O_4(\mathbb{C}) \).

By conjugation and by (2.1), we can assume that
\[
\psi \left( \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right) = \text{diag} (a^2, 1, 1, a^{-2}).
\]

The associated Langlands parameter \( \phi_\psi \) is, by definition, \( \phi_\psi: W_F \times SL_2(\mathbb{C}) \mapsto O_5(\mathbb{C}): \)
1. \( \phi_\psi|_{SL_2(\mathbb{C})} \equiv 1, \)
2. \( \phi_\psi(w) = \psi(w, \begin{pmatrix} |w|^\frac{1}{2} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix}) = \text{diag} (\mu(w)|w|, \mu(w), 1, \mu(w), \mu(w)|w|^{-1}). \)

The nontempered part of \( \phi_\psi \) is
\[
\phi_+(w) = \text{diag} (|w|, 1, 1, |w|^{-1}).
\]

Therefore \( \text{Cent} (im \phi_+, O_5(\mathbb{C})) = \mathbb{C}^\times \times O_3(\mathbb{C}) = M^* \) with \( M = F^\times \times SL_2(F) \), the Levi subgroup of the non-Siegel parabolic subgroup of \( \text{Sp}_4 \).
The tempered part of $\phi_\psi$ is
\[ \phi_\psi^0(w) = \text{diag}(\mu(w), \mu(w), 1, \mu(w), \mu(w)) \in M^*. \]

Therefore
\[ \text{Cent}(i\phi_\psi, O_5(\mathbb{C})) = \text{Cent}(i\phi_\psi^0, M^*) = \mathbb{C}^* \times O_2(\mathbb{C}) \times \{\pm 1\}. \]

So
\[ \text{Cent}(i\phi_\psi, O_5(\mathbb{C}))/Z_G \cdot \text{Cent}(i\phi_\psi, O_5(\mathbb{C})) = \{\pm 1\}. \]

For each place $v$, we can see that $\phi_\psi^0$ is the Langlands parameter for tempered representations $\{\pi_\pm(\mu)\}$, where $\pi_\pm(\mu)$ is the irreducible constituents of $\text{Ind}_B^G \chi(\mu, \mu)$ and $\phi_\psi$ is the Langlands parameter for the Langlands’ quotients $\{J_\pm(\mu_v)\}$ of $\text{Ind}_B^G \pi_\pm(\mu) \otimes e^{(e_1, H_{\psi})}$ (see [12] for more details).

Now we calculate $S_\psi = \text{Cent}(i\phi_\psi, O_5(\mathbb{C}))$. Let $u$ be a distinguished unipotent element with Jordan block $(3,1)$ in $O_4(\mathbb{C})$. Since $\mu$ is a nontrivial quadratic grössencharacter, $S_\psi^0 = 1$ and
\[ S_\psi^0/Z_G \cdot S_\psi^0 = \text{Cent}(u, O_4(\mathbb{C}))/\text{Cent}(u, O_4(\mathbb{C})) = \{\pm 1\}. \]

Here $\text{Cent}(u, O_4(\mathbb{C}))/\text{Cent}(u, O_4(\mathbb{C})) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Among the remaining automorphic forms in the Arthur packet of $\psi$ are the cuspidal representations studied by Howe and Piatetski-Shapiro (cf. [1], example 2.4.2).

This parametrizes the residual spectrum of $\text{Sp}_4$ obtained in [12], where Kim obtained this as the residue of the Eisenstein series associated to $\chi = \chi(\mu, \mu)$ at $\Lambda_0 = e_1$. It was also proved there that quadratic unipotent Arthur parameters exhaust the whole residual spectrum of $\text{Sp}_4$, coming from Borel subgroups. We observe that since $O_4(\mathbb{C})$ has only one distinguished unipotent orbit, namely the orbit $(3,1)$, our result covers the result of [12], coming from this conjugacy class, except for the constants.

5.2. One extreme case. We give an example which was our motivation for the general result: Let $G = \text{Sp}_{4n}$ and $\chi = \chi(\mu_1, \ldots, \mu_n, \mu_1, \ldots, \mu_n)$, where $\mu_i$’s are mutually distinct and quadratic. Then the Eisenstein series has a pole at $\Lambda_0 = e_1 + \cdots + e_n$ and the residue is given by
\[ \sum_{d \in D} (*) R(dc_1 \ldots c_n, \Lambda_0, \chi) \prod_{i=n+1}^{2n} (1 + R(c_i, \Lambda_0, \chi))^f, \]

where $D$ is the set of permutations $s$ which satisfy $s(i) < s(i + n)$ for $i = 1, \ldots, n$. 

Remark 3. The above technique can be used for $\text{GL}_n$ to prove that the Eisenstein series associated to $\chi = \chi(\mu, \ldots, \mu, \nu_1, \ldots, \nu_{r_1})$ has no pole of order $n-1$ if $\mu$ and $\nu_i$ are distinct. This is a very special case of the remarkable result proved by Moeglin and Waldspurger [24].

We divide $\Phi^+$ as follows:

$$
\Phi_1 : \quad e_i - e_j, \quad 1 \leq i < j \leq r_1 \\
\Phi_2 : \quad e_{r_1+i} - e_{r_1+j}, \quad 1 \leq i < j \leq r_2 \\
\Phi_3 : \quad e_i - e_{r_1+j}, \quad i = 1, \ldots, r_1, \quad j = 1, \ldots, r_2
$$

For $\alpha \in \Phi_3$, $\chi \circ \alpha^\vee$ is nontrivial. Let $W_i$ be the Weyl group associated to $\Phi_i$ for $i = 1, 2$. We apply Proposition 4.3 to $\theta = \Delta - \{e_{r_1} - e_{r_1+1}\}$. Then $W_0 = W_1 W_2$. Let $D$ be the set of distinguished coset representatives for $W_1 W_2$. Then as in the proof of Proposition 4.6, we can show that the constant term of the Eisenstein series $E_0(g, f, \Lambda)$ attached to $f \in I(A, \chi)$ has at most poles of order $< n-1$.


