EXAMPLES OF SECTION 6.5

Example 1. A 16 lb object stretches a spring \( \frac{8}{9} \) ft by itself. There is no damping and no external forces acting on the system. The spring is initially displaced 6 inches upwards from its equilibrium position and given an initial velocity of 1 ft/sec downward. Find the displacement \( u(t) \) at any time \( t \).

Solution. We first need to set up the equation for the problem. This requires us to find \( m \) and \( k \).

The mass is

\[
m = \frac{\text{Weight}}{g} = \frac{16}{32} = \frac{1}{2}.
\]

Now we find \( k \) by Hook’s law:

\[
k = \frac{mg}{L} = \frac{16}{\frac{8}{9}} = 18.
\]

We can now set up the IVP.

\[
u'' + 36u = 0, \quad u(0) = -\frac{1}{2}, \quad u'(0) = 1.
\]

For the initial conditions recall that upward displacement/motion is negative while downward displacement/motion is positive. Also, since we decided to do everything in feet we had to convert the initial displacement to feet.

The associated polynomial differential equation is

\[
(D^2 + 36)u = 0,
\]

which gives us the complex conjugate roots \( r = \pm 6i \). The general solution can be found to be:

\[
u = C_1 \cos(6t) + C_2 \sin(6t).
\]

To find the undetermined constants, we plug in the initial values:

\[
u(0) = C_1 = -\frac{1}{2},
\]

and

\[
u'(0) = -6C_1 \cos(0) + 6C_2 \sin(0) = 1 \implies C_2 = \frac{1}{6}.
\]

The displacement at any time \( t \) now is given by

\[
u(t) = -\frac{1}{2} \cos(6t) + \frac{1}{6} \sin(6t).
\]
Now, let’s convert this to a single cosine. First let’s get the amplitude, \( R \).

\[
R = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{6}\right)^2} = \frac{\sqrt{10}}{6}.
\]

The phase is now given by

\[
\phi = \cos^{-1}\left(-\frac{1/2}{\sqrt{10/6}}\right) = \cos^{-1}\left(-\frac{3}{\sqrt{10}}\right).
\]

This angle is between \( \pi/2 \) to \( \pi \), which makes \( \sin \phi > 0 \). So we are good, since we expect \( \sin \phi = \frac{1/6}{\sqrt{10/6}} = \frac{1}{\sqrt{10}} \). This yields

\[
u(t) = \frac{\sqrt{10}}{6} \cos(6t - \phi).
\]

Remark 2. The difficulty of this kind of problems is to determine the phase angle. For example, if

\[
u(t) = -\frac{1}{2} \cos(6t) - \frac{1}{6} \sin(6t).
\]

We still get the same amplitude \( R = \frac{\sqrt{10}}{6} \). So the angle

\[
\phi = \cos^{-1}\left(-\frac{1/2}{\sqrt{10/6}}\right) = \cos^{-1}\left(-\frac{3}{\sqrt{10}}\right)
\]

remain the same as in the example, and thus is between \( \pi/2 \) to \( \pi \). However, in this case our \( \sin \phi = \frac{1/6}{\sqrt{10/6}} = -\frac{1}{\sqrt{10}} < 0 \). A contradiction. An angle that works will be

\[
\theta = 2\pi - \phi,
\]

as cosine is an even function

\[
\cos(\theta) = \cos(2\pi - \phi) = \cos(-\phi) = \cos(\phi) = -\frac{3}{\sqrt{10}}
\]

and sine is an odd function

\[
\sin(\theta) = \sin(2\pi - \phi) = \sin(-\phi) = -\sin(\phi) = -\frac{1}{\sqrt{10}}.
\]