EXAMPLES OF SECTION 6.9

Example 1. Find a general solution to the following differential equation.
\[ t^2y'' + 2ty' - 2y = 0 \]
given that \(y_1(t) = t\) is a solution.

Solution. We look for solutions of the form \(y_2(t) = v(t)y_1(t) = tv(t)\). Plugging \(y_2\) into the differential equation yields
\[ t^2[2v' + tv''] + 2t[v + tv'] - 2tv = 0. \] (1)
Here we have used the following fact
\[ (uv)'' = u'' + 2u'v' + v''. \]
Now because \(y_1\) is a solution to the original differential equation, we just ignore all the terms in (1) containing \(v\). This gives
\[ t^3v'' + 4t^2v' = 0. \] (2)
Let \(w = v'\). (2) becomes
\[ t^3w' + 4t^2w = 0. \]
This is a separation of variables kind of equation. Solving it, we obtain
\[ w = c_2t^{-4}. \]
Integrating this results gives
\[ v = c_2t^{-3} + c_1, \]
which yields
\[ y_2(t) = c_1t + c_2t^{-2}. \]