EXAMPLES OF SECTIONS 1.8

Question 1. Solve
\[
\begin{aligned}
y'x + (2x - 3)y &= 5x^5y^4, \\
y(1) &= 1.
\end{aligned}
\]

Question 2. Solve
\[
(x^4 - 2t^3)dt + (t^4 - 2tx^3)dx = 0.
\]

Solutions.

1. Write the equation as
\[
y' + \frac{2x - 3}{x}y = 5x^4y^4,
\]
for \( x \neq 0 \), which is a Bernoulli equation with \( n = 4 \). Set \( v = y^{1-n} = y^{-3} \).
The equation for \( v \) then becomes
\[
\frac{dv}{dx} + (1 - 4)\frac{2x - 3}{x}v = (1 - 4)5x^4,
\]
or
\[
\frac{dv}{dx} + \left(\frac{9}{x} - 6\right)v = -15x^4.
\]
Using the formula for first order linear D.E.‘s with \( p(x) = \frac{9}{x} - 6 \) and \( q(x) = -15x^4 \), we have
\[
e^{-\int p(x) \, dx} = e^{6x - 9\ln x} = x^{-9}e^{6x},
\]
\[
e^{\int p(x) \, dx} = e^{-6x + 9\ln x} = x^9e^{-6x}.
\]
where we assumed \( x > 0 \) since the problem is defined only for \( x > 0 \) or \( x < 0 \) (because \( x \neq 0 \)). From this we get
\[
\int q(x)e^{\int p(x) \, dx} \, dx = -15 \int x^{13}e^{-6x} \, dx.
\]
This integral is done by a tiresome (but not difficult) process of integrating by parts thirteen times. The answer is
\[
- \frac{e^{-6x}}{314928} \left( 25025 + 150150x + 450450x^2 + 900900x^3 + 1351350x^4 + 1621620x^5 + 1621620x^6 \\
+ 1389960x^7 + 1042470x^8 + 694980x^9 + 416988x^{10} + 227448x^{11} + 113724x^{12} + 52488x^{13} \right)
\]
Denote the above expression by \( f(x) \). Then using the formula for solutions of first order linear equations,

\[
v(x) = x^{-9}e^{6x}(f(x) + C),
\]

from which follows

\[
y(x) = \left[x^{-9}e^{6x}(f(x) + C)\right]^{-\frac{1}{3}}.
\]

To find \( C \) use \( y(1) = 1 \),

\[
y(1) = 1 = \left[e^{6\left(f(1) + C\right)}\right]^{-\frac{1}{3}} \Rightarrow C = e^{-6} - f(1),
\]

thus

\[
y(x) = \left[x^{-9}e^{6x}(f(x) + e^{-6} - f(1))\right]^{-\frac{1}{3}}.
\]

2. We divide both sides of the equation by \( t^4 \) and transform it into

\[
\frac{dx}{dt} = \frac{(x/t)^4 - 2x/t}{1 - 2(x/t)^3}.
\]

This is a homogeneous equation and thus we can make the substitution \( V(t) = x/t \).

\[
tV' + V = \frac{V^4 - 2V}{1 - 2V^3} \Rightarrow tV' = 3 \frac{V^4 - V}{1 - 2V^3} \Rightarrow \int \frac{1 - 2V^3}{V^4 - V} \, dV = \int \frac{3}{t} \, dt + C.
\]

Using partial fraction and noticing that \( V^4 - V = V(V - 1)(V^2 + V + 1) \),

we have

\[
\frac{1 - 2V^3}{V^4 - V} = \frac{1}{V} - \frac{1}{3V - 1} - \frac{1}{3V^2 + V + 1}.
\]

This yields

\[
\ln(|V|\sqrt[3]{V - 1}\sqrt{V^2 + V + 1}) = -9 \ln |t| + C.
\]