EXAMPLES OF SECTIONS 1.9

Question 1. Solve  \( \frac{y}{x} + 6 + (\ln|x| - 2)y' = 0 \).

Question 2. Find  \( b \) such that  \((xy^2 + bx^2y)dx + (x + y)x^2dy\) is exact and solve the equation.

Question 3. Find the integrating factor for  \((3x^3 + y)dx + (2x^2y - x)dy = 0\) and solve the equation.

Solutions.
1. First, one checks

   \[ \frac{\partial}{\partial y} M = \frac{1}{x} = \frac{\partial}{\partial x} N, \]

   and thus this equation is exact. Set

   \[ \phi(x, y) = \int M(x, y) \, dx + h(y) = y \ln |x| + 3x^2 + h(y), \]

   and differentiate \( \phi \) w.r.t. \( y \)

   \[ \frac{\partial}{\partial y} \phi(x, y) = x^2 + h'(y) = N(x, y) = \ln |x| - 2. \]

   We thus obtain \( h'(y) = -2 \), which yields \( h(y) = -2y \). In summary,

   \[ \phi(x, y) = y \ln |x| + 3x^2 - 2y = C, \]

   or equivalently

   \[ y(x) = \frac{C - 3x^2}{\ln |x| - 2} \]

   is the general solution.

2. We need

   \[ \frac{\partial}{\partial y} M = 2xy + bx^2 = \frac{\partial}{\partial x} N = 3x^2 + 2xy, \]

   which clearly implies \( b = 3 \). As above,

   \[ \phi(x, y) = \int M(x, y) \, dx + h(y) = \frac{1}{2}x^2y^2 + x^3y + h(y) \]

   \( \Rightarrow \frac{\partial}{\partial y} \phi(x, y) = x^2y + x^3 + h'(y) = x^3 + x^2y \)

   \( \Rightarrow h'(y) = 0 \Rightarrow h(y) = 0 \)

   \( \Rightarrow \phi(x, y) = \frac{1}{2}x^2y^2 + x^3y = C. \)
3. \((3x^3 + y)dx + (2x^2y - x)dy = 0\) Let \(M(x, y) = 3x^3 + y\) and \(N(x, y) = 2x^2y - x\). Then

\[
\partial_y M - \partial_x N = 2(1 - 2xy) \implies \frac{\partial_y M - \partial_x N}{N} = -\frac{2}{x}.
\]

Then

\[
I(x) = e^{\int \frac{2}{x} \, dx} = \frac{1}{x^2}.
\]

The equation

\[(3x - \frac{y}{x^2})dx + (2y - \frac{1}{x})dy = 0\]

is exact. Following the same procedure as above, we get

\[
\phi(x, y) = \frac{3}{2}x^2 + y^2 - \frac{y}{x} = C
\]

is the general solution.