Partial answers to some problems in the practice exam

Warning: I did not have time to double check the answers!

Warning: The problems 1 and 2 are modified, please replace the first page with the new one on the web!

1. \[(n + 2)a_{n+2} - a_{n+1} + a_n = 0, \quad n = 0, 1, 2, \cdots\]

2. \[(n + 2)(n + 1)a_{n+2} + (n + 1)(1 - n)a_{n+1} + a_n = 0, \quad n = 0, 1, 2, \cdots\]

3. \(x_0 = 2\): regular singular point; \(x_0 = -1\): irregular singular point.

4. \(y(x) = \frac{5}{3}x - \frac{2}{3}x^{-\frac{1}{2}}\)

5. (c) and (c)

6. Critical values: \(\alpha = 0\) and \(\alpha = \pm 2\sqrt{3}\).

7. \(X(t) = \bar{e}^{-t} + \bar{b}t + \bar{c}\)

8. \(X(t) = \bar{e}^{t} + \bar{b}\). Then determine \(\bar{a}\) and \(\bar{b}\) from the equation.

9. \[\ddot{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}\]
   and determine \(c_1\) and \(c_2\) from the initial conditions.

10. \(0 < \gamma < 4\sqrt{2}\)

11.

12.

13. \(\lambda = (n + \frac{1}{2})^2, \cos(n + \frac{1}{2})x, \quad n = 0, 1, 2, \cdots\)

14. \(\lambda = \left(\frac{n+\frac{1}{2}}{2}\right)^2, \sin\left(\frac{n+\frac{1}{2}}{2}\pi x\right), \quad n = 0, 1, 2, \cdots\)

15. \[f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} (1 - \cos \frac{n\pi}{3}) \sin \frac{n\pi x}{3}\]

16. \[f(x) = 1 - \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin \frac{n\pi}{2} \cos \frac{n\pi x}{2}\]
17. \[ f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} (-1)^{n+1} \sin n\pi x \]

18. \[ 2X''(x) + 3x X'(x) + \lambda X(x) = 0, \ X(0) = X'(1) = 0 \]

19. Heat equation with \( \alpha = 1 \).

20. \[ u(x, t) = \sum_{n=1}^{\infty} \frac{2}{n\pi} (-1)^n \sin n\pi x e^{-\left(2\pi\right)^2 t} + 2x \]

21. \[ u(x, t) = \sum_{n=1}^{\infty} \frac{8}{n^2\pi^2} \sin \frac{n\pi x}{2} \sin \frac{n\pi x}{2} \cos \frac{2n\pi t}{2} \]

22. \[ u(x, y) = \left( \sinh \frac{n\pi}{2} \right)^{-1} \left( \sinh 2\pi x \sin 2\pi y + \sinh \pi x \sin \pi y \right) \]